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Perturbative Quantum Field Theory in the String-Inspired Formalism

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Abstract

We review the status and present range of applications of the "string – inspired" approach to perturbative quantum field theory. This formalism offers the possibility of computing effective actions and S-matrix elements in a way which is similar in spirit to string perturbation theory, and bypasses much of the apparatus of standard second-quantized field theory. Its development was initiated by Bern and Kosower, originally with the aim of simplifying the calculation of scattering amplitudes in quantum chromodynamics and quantum gravity. We give a short account of the original derivation of the Bern-Kosower rules from string theory. Strassler's alternative approach in terms of first-quantized particle path integrals is then used to generalize the formalism to more general field theories, and, in the abelian case, also to higher loop orders. A considerable number of sample calculations are presented in detail, with an emphasis on quantum electrodynamics.

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Contents

1. Introduction: Strings vs. Particles, First vs. Second Quantization

One of the main motivations for the study of string theory is the fact that it reduces to quantum field theory in the limit where the tension along the string becomes infinite [1,2,3,4]. In this limit all massive modes of the string get suppressed, and one remains with the massless modes. Those can be identified with ordinary massless particles such as gauge bosons, gravitons, or massless spin-$\frac{1}{2}$ fermions. Moreover, the interactions taking place between those massless modes turn out to be familiar from quantum field theory. In particular, it came as a pleasant surprise that a consistent theory of fundamental strings must by necessity include quantum gravity [5].

Regardless of whether string theory is realized in nature or not, those mathematical facts already lend us a new perspective on quantum field theory, which we now find embedded in a theory which is not only vastly more complex, but also structurally different. In particular, a major difference between string theory and particle theory is that, in string theory, the full perturbative S-matrix is calculable in first quantization using the Polyakov path integral, which describes the propagation of a single string in a given background. An adequate second quantized field theory for strings was built after considerable efforts [6,7], allowing one to define off-shell amplitudes with the correct factorization properties, and even to compute nonperturbative effects in string theory [8,9,10]. Nevertheless, so far the second quantized approach has not led to improvements over the first quantized formalism as far as the calculation of perturbative string scattering amplitudes is concerned.

In ordinary quantum field theory, of course, perturbative calculations are usually performed using second quantization, and Feynman diagrams. First quantized alternatives have been developed already at the very inception of relativistic quantum field theory [11], and will, in fact, be the main subject of the present review. However it appears that, until recently, they were hardly ever seriously considered as an efficient tool for standard perturbative calculations. This discrepancy between string and particle theory becomes something of a puzzle as soon as one considers the latter as a limiting special case of the former. And we would like to convince the reader in the following that this apparent paradox owes more to historical development than to mathematical fact.

If string theory reduces to field theory in the infinite string tension limit, then clearly it should be possible to obtain S-matrix elements in certain field theory models by analyzing the corresponding string scattering amplitudes in this limit. It goes without saying that the calculation of string amplitudes is generally much more difficult than the calculation of the corresponding field theory amplitudes, so that the practical value of such a procedure may appear far from obvious. Nevertheless, it turns out to be sufficiently motivated by the different organization of both types of amplitudes, and by the different methods available for their computation.

As early as 1972 Gervais and Neveu observed that string theory generates Feynman rules for Yang-Mills theory in a special gauge that has certain calculational advantages [12]. At the loop level, the first explicit calculation along these lines was performed in 1982 by Green, Schwarz and Brink, who obtained the one − loop 4 − gluon amplitude in $N = 4$ Super Yang-Mills theory from the low energy limit of superstring theory [13]. However, serious interest in this subject commenced only in 1988, when it was shown by several authors that the one-loop β − function for Yang-Mills theory can be extracted from the genus one partition function of an open string coupled to a background gauge field [14,15,16,17]. This calculation gives also some insight into the well-known fact that this β − function coefficient vanishes for $D = 26$, the critical

dimension of the bosonic string, when calculated in dimensional regularization [18].

A systematic investigation of the infinite string tension limit was undertaken in the following years by Bern and Kosower [19,20,21]. This was done again with a view on application to non-abelian gauge theory, however now to the computation of complete on-shell scattering amplitudes. Again the idea was to calculate, say, gauge boson scattering amplitudes in an appropriate string model containing $SU(N_c)$ gauge theory, up to the point where one has obtained an explicit parameter integral representation for the amplitude considered. At this stage one performs the infinite string tension limit, which should eliminate all contributions due to propagating massive modes, and lead to a parameter integral representation for the corresponding field theory amplitude.

In the present work, we will concentrate on a different and more elementary approach to this formalism, which does not rely on the calculation of string amplitudes any more, and uses string theory only as a guiding principle. Only a sketchy account will therefore be given of string perturbation theory, and the reader is referred to the literature for the details [5,22].

The basic tool for the calculation of string scattering amplitudes is the Polyakov path integral. In the simplest case, the closed bosonic string propagating in flat spacetime, this integral is of the form

$$\Big\langle\ V_1 \cdots V_N \Big\rangle \sim \sum_{\rm top} \int \mathcal{D}h \int \mathcal{D}x(\sigma,\tau) V_1 \cdots V_N \, \mathrm{e}^{-S[x,h]} \tag{1.1}$$

This path integral corresponds to first quantization in the sense that it describes a single string propagating in a given background. The parameters σ, τ parametrize the world sheet surface swept out by the string in its motion, and the integral $\int \mathcal{D}x(\sigma,\tau)$ has to be performed over the space of all embeddings of the string world sheet with a fixed topology into spacetime. The integral $\int \mathcal{D}h$ is over the space of all world sheet metrics, and the sum over topologies $\sum_{\rm top}$ corresponds to the loop expansion in field theory (fig. 1).

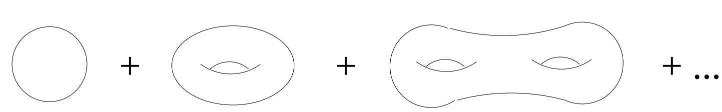


Figure 1: The loop expansion in string perturbation theory.

If the closed string is assumed to be oriented, there is only one topology at any fixed order of loops.

In the case that the background is simply Minkowski spacetime the world sheet action is given by

$$S[x,h] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{h} h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu \tag{1.2}$$

where $\frac{1}{2\pi\alpha'}$ is the string tension. Note that in Polyakov's formulation the action is quadratic in the coordinate field x.

The vertex operators $V_1, \ldots, V_N$ represent the scattering string states. In the case of the open string, which is the more relevant one for our purpose, the world sheet has a boundary, and the vertex operators are inserted on this boundary. For instance, for the open oriented string at the one-loop level the world sheet is just an annulus, and a vertex operator may be integrated along either one of the two boundary components (fig. 2).

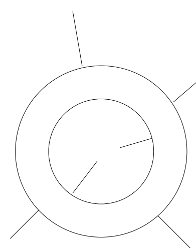


Figure 2: Vertex operators inserted on the boundary of the annulus.

The vertex operators most relevant for us are of the form

$$V^\phi[k] = \int d\tau \, \mathrm{e}^{ik\cdot x(\tau)} \tag{1.3}$$

$$V^A[k,\varepsilon,a] = \int d\tau \, T^a \varepsilon \cdot \dot{x}(\tau) \, \mathrm{e}^{ik\cdot x(\tau)} \tag{1.4}$$

They represent a scalar and a gauge boson particle with definite momentum k and polarization vector ε. T^a is a generator of the gauge group in some representation. The integration variable τ parametrizes the boundary in question. Since the action is Gaussian, $\int \mathcal{D}x$ can be performed by Wick contractions,

$$\left\langle x^\mu(\tau_1) x^\nu(\tau_2) \right\rangle = G(\tau_1,\tau_2)\, \eta^{\mu\nu} \tag{1.5}$$

G denotes the Green's function for the Laplacian on the annulus, restricted to its boundary, and $\eta^{\mu\nu}$ the Lorentz metric.

In $D = 26$, the critical dimension of the bosonic string, the Polyakov path integral is conformally invariant. The remaining path integral over the infinite dimensional space of all world sheet metrics h can then be reduced to the space of conformal equivalence classes, which is finite dimensional. The actual integration domain, moduli space, is somewhat smaller, since a further discrete symmetry group has to be taken into account. At this stage, then, the amplitude is in a form suitable for performing the infinite string tension limit. It turns out that, in this limit, only certain corners of the whole moduli space contribute. The amplitude thus splinters into a

number of pieces, which individually are parameter integrals of the same type encountered in field theory Feynman diagram calculations.

For illustration, consider the following two-point "Feynman diagram" for the closed string (fig. 3).

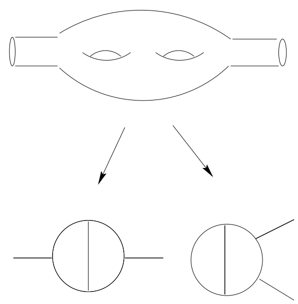


Figure 3: Infinite string tension limit of a string diagram.

In the $\alpha' \rightarrow 0$ limit this Riemann surface gets squeezed to a Feynman graph, although not to a single one; two Feynman diagrams of different topologies emerge. This proliferation becomes, of course, much worse at higher orders. Moreover, in gauge theory or gravity it is further enhanced by the existence of quartic and higher order vertices, which lead to many more possible topologies. The generating string theories thus have a much smaller number of "Feynman diagrams" then the limiting field theories, which is another major motivation for the use of string-techniques in field theory.

The uses of the Polyakov path integral are not restricted to the calculation of scattering amplitudes. As pointed out by Fradkin and Tseytlin [23], it is equally useful for the calculation of string effective actions. For instance, an open oriented string propagating in the background of a Yang-Mills field A would generate an effective action for this background field given by the following modification of the Polyakov path integral,

$$
\begin{aligned}
\Gamma[A] &\sim \sum_{\text{top}} \int \mathcal{D}h \int \mathcal{D}x(\sigma,\tau)\, \mathrm{e}^{-S_0 - S_I} \\
S_0 &= -\frac{1}{4\pi\alpha'} \int_M d\sigma d\tau \sqrt{h} h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu \\
S_I &= \int_{\partial M} d\tau\, ie\, \dot{x}^\mu A_\mu(x(\tau))
\end{aligned}
\tag{1.6}
$$

The sum now extends over all oriented bounded manifolds. The free term S_0 is the same as above, eq.(1.2), and the interaction term S_I has to be integrated along all components of the boundary. For simplicity, we have written the interaction term for the abelian case; in the non-abelian case, a colour trace and path ordering would have to be included. This will be discussed later on in the field theory context.

Metsaev and Tseytlin calculated the one-loop path integral exactly for the constant field strength case [16], and verified that the $\alpha' \rightarrow 0$ limit coincides with the corresponding effective action in Yang-Mills theory. In particular, the correct β - function coefficient can be

read off from the $F_{\mu\nu}^2$ – term. This procedure is not completely rigorous, though, since the open bosonic string theory cannot be consistently truncated from 26 down to four dimensions. In their analysis of the N - gluon amplitude [21], Bern and Kosower therefore used, instead of the open string, a certain heterotic string model containing $SU(N_c)$ Yang-Mills theory in the infinite string tension limit. This allows for a consistent reduction to four dimensions, at the price of a more complicated representation of this amplitude. By an explicit analysis of the infinite string tension limit, they succeeded in deriving a novel type of parameter integral representation for the on-shell N - gluon amplitude in Yang-Mills theory, at the tree- and one-loop level. Moreover, they established a set of rules which allows one to construct this parameter integral, for any number of gluons and choice of helicities, without referring to string theory any more.

While those rules are very different from the corresponding field theoretic Feynman rules, the precise connection and equivalence between both sets of rules were soon established [24]. Moreover, once an understanding of the rules had been reached, it emerged that the consistency requirements motivating the choice of the heterotic string were not really relevant in their derivation. An alternative derivation using the naive truncated open string was given [25], and even yielded a somewhat simpler set of rules.

This set of rules will be discussed in detail in chapter 2. For the moment, let us just mention some advantages of the "Bern-Kosower Rules" as compared to the Feynman rules:

1. Superior organization of gauge invariance.
2. Absence of loop momentum, which reduces the number of kinematic invariants from the beginning, and allows for a particularly efficient use of the spinor helicity method.
3. The method combines nicely with spacetime supersymmetry.
4. Calculations of scattering amplitudes with the same external states but particles of different spin circulating in the loop are more closely related than usual.

The last two points are, of course, closely related. The efficiency of these rules has been demonstrated by the first complete calculation of the one – loop five – gluon amplitude [26]. A similar set of rules for graviton scattering was derived from closed string theory in [27]. Those have been used for the first calculation of the complete one – loop four – graviton amplitude in quantum gravity [28].

Since the Bern-Kosower rules do not refer to string theory any more, the question naturally arises whether it should not be possible to re-derive them completely inside field theory. Obviously, such a re-derivation should be attempted starting from a first-quantized formulation of ordinary field theory, rather than from standard quantum field theory. As we mentioned in the beginning, such formulations have been known for decades, albeit only for a very limited number of models. Already in 1950, in the appendix A of his famous paper "Mathematical Formulation of the Quantum Theory of Electromagnetic Interaction" [11], Feynman presented such a formalism for the case of scalar quantum electrodynamics, "for its own interest as an alternative to the formulation of second quantization". What he states here is that the amplitude for a charged scalar particle to move, under the influence of the external potential A_μ, from point x_μ to x'_μ in Minkowski space is given by

$$\int_0^\infty ds \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x(\tau)\, \exp\Bigl(-\frac{1}{2}im^2 s\Bigr) \exp\Bigl[-\frac{i}{2}\int_0^s d\tau \bigl(\frac{dx_\mu}{d\tau}\bigr)^2 - i\int_0^s d\tau \frac{dx_\mu}{d\tau} A_\mu(x(\tau))$$
$$-\frac{i}{2}e^2 \int_0^s d\tau \int_0^s d\tau' \frac{dx_\mu}{d\tau}\frac{dx_\nu}{d\tau'}\delta_+^{\mu\nu}(x(\tau)-x(\tau'))\Bigr] \tag{1.7}$$

That is, for a fixed value of the variable s (which can be identified with Schwinger proper time) one can construct the amplitude as a certain quantum mechanical path integral. This path integral has to be performed on the set of all open trajectories running from x to x' in the fixed proper time s. The action consists of the familiar kinetic term, and two interaction terms. Of those the first represents the interaction with the external field, to all orders in the field, while the second one describes an arbitrary number of virtual photons emitted and re-absorbed along the trajectory of the particle (δ_+ denotes the photon propagator). In second quantized field theory, this amplitude would thus correspond to the infinite sequence of Feynman diagrams shown in fig. 4.

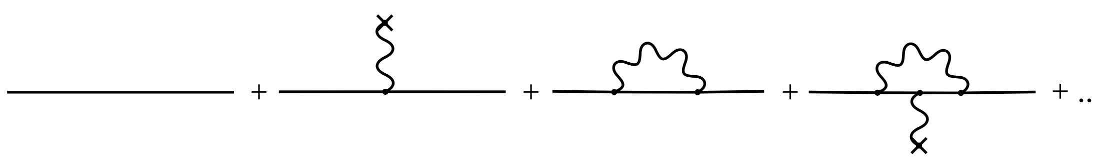


Figure 4: Sum of Feynman diagrams represented by a single path integral.

As Feynman proceeds to show, this representation extends in an obvious way to the case of an arbitrary fixed number of scalar particles, moving in an external potential and exchanging internal photons, and thus to the complete S-matrix for scalar quantum electrodynamics. Every scalar line or loop is then separately described by a path integral such as the one above. The path integrals are coupled by an arbitrary number of photon insertions. The derivation of this type of path integral will be discussed in detail in chapter 3.

In the present work, we are mainly concerned with path integrals for closed loops. Let us therefore rewrite Feynman's formula for the case of a single closed loop in the external field, with no internal photon corrections. What we have at hand then is simply a representation of the one-loop effective action for the Maxwell field [2] ,

$$\Gamma[A] = \int_0^\infty \frac{dT}{T}\, \mathrm{e}^{-m^2 T} \int \mathcal{D}x \exp\left[-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ieA_\mu \dot{x}^\mu\right)\right] \tag{1.8}$$

The path integral runs now over the space of closed trajectories with period T, $x^\mu(T) = x^\mu(0)$.

[2] The proper time parameter s has been rescaled and Wick rotated, $s \to -i2T$. The spacetime metric will also be taken as Euclidean, except when stated otherwise (upper and lower indices will be used purely for typographical convenience). Moreover, we anticipate dimensional regularization and thus usually continue to D Euclidean dimensions.

The comparison of the Fradkin-Tseytlin path integral (1.6) with the path integral (1.8) shows that the former is clearly a string theoretic generalization of the latter. Conversely, at least at the one-loop level it is not difficult to show that the Feynman path integral is precisely the infinite string tension limit of the Fradkin-Tseytlin path integral. Naively, one can think of the annulus in fig. 2 being squeezed to its boundary.

The path integral representation eq.(1.8) generalizes in various ways to spinor quantum electrodynamics. In the fermion loop case, one has a basic choice to make in the treatment of the spin degrees of freedom. Those can be incorporated either by explicit γ - matrices [29,30], or by Grassmann variables [31,32,33,34], which carry the same algebraic properties. The first, "bosonized", version may be preferable for certain purposes such as the evaluation of path integrals by numerical or saddle point approximation. Nevertheless, we will generally use the second alternative, since it offers the possibility to use worldline supersymmetry in a computationally meaningful way.

Supersymmetric worldline Lagrangians were constructed soon after the advent of supersymmetry. This led to an intensive study of field theories in $1+0$ dimensions, and to the discovery that the worldline Lagrangian appropriate for the description of a Dirac particle is precisely the one for $N=1$ supergravity [35,36]. As a consequence of that work, the generalization of our path integral eq.(1.8) for the one-loop effective action to spinor QED can be written as a super path integral [31,37,38,39]

$$
\begin{aligned}
\Gamma[A] &= -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int_P \mathcal{D}x \int_A \mathcal{D}\psi \\
&\quad \times \exp\Bigl[-\int_0^T d\tau \Bigl(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\cdot\dot{\psi} + ieA_\mu \dot{x}^\mu - ie\psi^\mu F_{\mu\nu}\psi^\nu\Bigr)\Bigr]
\end{aligned}
\tag{1.9}
$$

In addition to the integral over the periodic functions $x^\mu(\tau)$, we have now a second path integral over the functions $\psi^\mu(\tau)$, which are Grassmann valued and antiperiodic (the periodicity properties are expressed by the subscripts P, A on the path integral). The global minus sign accounts for the Fermi statistics of the spinor loop.

Comparing this formula with eq.(1.8), it becomes immediately clear that one can think of this double path integral as breaking up the Dirac spinor into an "orbital part" and a "spin part". The former is represented by the same coordinate path integral as the scalar particle, the latter by the additional Grassmann path integral.

In writing this path integral, we have already gauge-fixed the local one-dimensional supergravity. This leaves over a global supersymmetry, "worldline supersymmetry",

$$
\begin{aligned}
\delta x^\mu &= -2\eta\psi^\mu \\
\delta\psi^\mu &= \eta\dot{x}^\mu
\end{aligned}
\tag{1.10}
$$

with a constant Grassmann parameter η. As we will see later on, the existence of this symmetry has far-reaching calculational consequences.

A vast amount of literature is available on this type of relativistic particle Lagrangians, and the corresponding path integrals. However, only a minor part of it is concerned with attempts to use

them as a tool for actual calculations in quantum field theory. Much of particularly the early literature emphasizes the one-dimensional over the spacetime point of view, or is concerned with the formal properties of such worldline field theories. In particular, one-dimensional field theories are often used for a comparative study of the various known quantization procedures (see, e.g., [40]).

Of those applications which have come to this author's notice, let us mention the work of Halpern et al. [41,42], who proposed to use first quantized path integrals for a construction of the strong-coupling expansion in non-abelian gauge theory. More recently, various attempts have been made to apply worldline path integrals to nonperturbative calculations, using various approximation schemes for the path integral. See, e.g., [43] for scalar field theory, [44] for heavy-meson theory, [45,46] for QCD, and [47] for meson-nucleon theory applications. Some applications to QED can be found in [48,49,38,50,51,52,53], to statistical physics in [54].

Probably best-known is, however, the application to the calculation of anomalies and index densities [55,56,57,58,59,60,61,62]. Here a number of special cases of the Atiyah-Singer index theorem could be reproduced in an elementary way by rewriting supertraces of heat kernels for the corresponding operators in terms of supersymmetric particle path integrals [56,57,58]. Nevertheless, despite this remarkable success it seems that, until recently, the first quantized formalism was never seriously considered as a competitor to the usual Feynman diagrammatic approach with regard to everyday life calculations of scattering amplitudes or effective actions.

The principle of how one might reproduce ordinary perturbation theory in the first quantized formalism, simply by mimicking string perturbation theory, was already sketched in chapter 9 of Polyakov's book [63]. However it was only after the work of Bern and Kosower, when it had become clear that techniques from first-quantized string perturbation theory *do* have the potential to improve on the efficiency of field theory calculations, that such an approach was seriously investigated by Strassler [64,65].

Let us demonstrate the method for the example of the scalar loop, eq.(1.8). The basic idea is simple: We will evaluate this path integral in precisely the same way as one calculates the Polyakov path integral in string theory, i.e. in a one-dimensional perturbation theory. If we expand the "interaction exponential",

$$\exp\Big[-\int_0^T d\tau\, ieA_\mu\dot x^\mu\Big] \quad = \quad \sum_{N=0}^{\infty}\frac{(-ie)^N}{N!}\prod_{i=1}^N\int_0^T d\tau_i\Big[\dot x^\mu(\tau_i)A_\mu(x(\tau_i))\Big] \tag{1.11}$$

the individual terms correspond to Feynman diagrams describing a fixed number of interactions of the scalar loop with the external field (fig. 5).
By standard field theory, the corresponding N − photon correlator is then obtained by specializing to a background consisting of a sum of plane waves with definite polarizations,

$$A_\mu(x) = \sum_{i=1}^N \varepsilon_{i\mu}\mathrm{e}^{ik_i\cdot x} \tag{1.12}$$

and picking out the term containing every ε_i once (this also removes the $\frac{1}{N!}$ in eq.(1.11)). We find thus exactly the same photon vertex operator used in string perturbation theory, eq. (1.4), inserted on a circle instead on the boundary of the annulus.

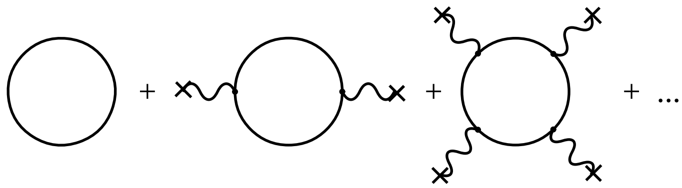


Figure 5: Expanding the path integral in powers of the background field.

At this stage the path integral has become Gaussian, which reduces its evaluation to the task of Wick contracting the expression

$$\Big\langle \dot{x}_1^{\mu_1} \mathrm{e}^{ik_1\cdot x_1} \cdots \dot{x}_N^{\mu_N} \mathrm{e}^{ik_N\cdot x_N} \Big\rangle \tag{1.13}$$

The Green's function to be used is now simply the one for the second-derivative operator, acting on periodic functions. To derive this Green's function, first observe that $\int \mathcal{D}x(\tau)$ contains the constant functions, which we must get rid of to obtain a well-defined inverse. Let us therefore restrict our integral over the space of all loops by fixing the average or "center of mass" position x_0^μ of the loop,

$$x_0^\mu \equiv \frac{1}{T}\int_0^T d\tau\, x^\mu(\tau) \tag{1.14}$$

For effective action calculations this reduces the effective action to the effective Lagrangian. In scattering amplitude calculations, the integral over x_0 just gives momentum conservation. The reduced path integral $\int \mathcal{D}y(\tau)$ over $y(\tau) \equiv x(\tau) - x_0$ has an invertible kinetic operator. The inverse is easily seen to be, up to an irrelevant constant,

$$2\langle \tau_1 \mid \Big(\frac{d}{d\tau}\Big)^{-2} \mid \tau_2\rangle = G_B(\tau_1,\tau_2) \tag{1.15}$$

with the "bosonic" worldline Green's function

$$G_B(\tau_1,\tau_2) = \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T} \tag{1.16}$$

(a "fermionic" worldline Green's function G_F will be introduced later on). For the performance of the Wick contractions, it is convenient to formally exponentiate all the $\dot{x}_i$'s, writing

$$\varepsilon_i \cdot \dot{x}_i\, \mathrm{e}^{ik_i\cdot x_i} = \mathrm{e}^{\varepsilon_i\cdot\dot{x}_i + ik_i\cdot x_i} \mid_{\mathrm{lin}(\varepsilon_i)} \tag{1.17}$$

This allows one to rewrite the product of N photon vertex operators as an exponential. Then one needs only to "complete the square" to arrive at the following closed expression for the one-loop N - photon amplitude in scalar QED,

$$\Gamma_{\rm scal}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\prod_{i=1}^N\int_0^T d\tau_i$$
$$\times\exp\Bigl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}G_{Bij}k_i\cdot k_j - i\dot G_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\Bigr]\Bigr\}\,|_{\rm multi-linear} \tag{1.18}$$

Here it is understood that only the terms linear in all the $\varepsilon_1,\ldots,\varepsilon_N$ have to be taken. Besides the Green's function G_B also its first and second derivatives appear,

$$\begin{aligned}\dot G_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)-2\frac{(\tau_1-\tau_2)}{T}\\ \ddot G_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2)-\frac{2}{T}\end{aligned} \tag{1.19}$$

Dots generally denote a derivative acting on the first variable, $\dot G_B(\tau_1,\tau_2)\equiv\frac{\partial}{\partial\tau_1}G_B(\tau_1,\tau_2)$, and we abbreviate $G_{Bij}\equiv G_B(\tau_i,\tau_j)$ etc. The factor $[4\pi T]^{-\frac{D}{2}}$ represents the free Gaussian path integral determinant factor.

The expression (1.18) which we have arrived at in this quite elementary way is identical with the "Bern-Kosower Master Formula" for the special case considered [21,66]. We will discuss various generalizations and applications of this formula later on. For now, the important point to note is that we have at hand here a single unifying generating functional for the one-loop photon S-matrix – something for which no known analogue exists in standard field theory.

How does this master integrand relate to the integrals appearing in an ordinary Feynman parameter calculation of this amplitude? Note that in (1.18) every photon leg is integrated around the loop independently. As we will see in detail later on, once one restricts the integration domain to a fixed ordering $\tau_{i_1}>\tau_{i_2}>\cdots>\tau_{i_N}$, it is not difficult to identify the integrand with the corresponding Feynman parameter integral. In particular, there is an exact correspondence between the δ – function appearing in the second derivative of G_B, and the seagull–vertex of scalar quantum electrodynamics. However the complete integral does not represent any particular Feynman diagram, with a fixed ordering of the external legs, but *the sum of them* (fig. 6):

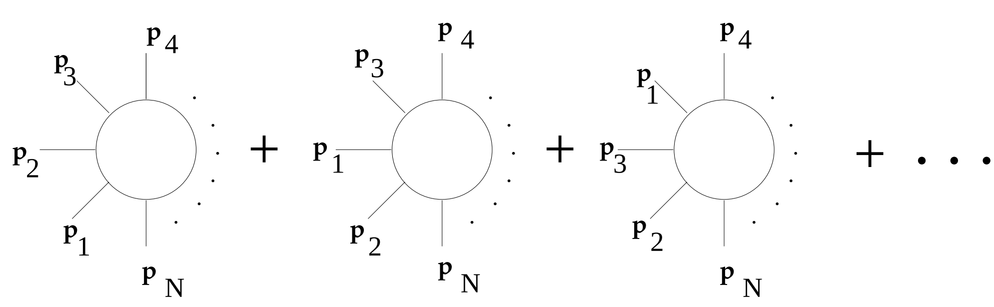


Figure 6: Sum of one-loop diagrams with permuted legs.

This fact may not seem particularly relevant at the one-loop level. However it is important to note that the path integral representation eq.(1.8) and the resulting integral representation eq.(1.18) are valid off-shell [3]. We can therefore use this formula to sew together a pair of legs, say, legs number 1 and N, and obtain a parameter integral representing the complete two-loop $(N-2)$ – photon amplitude (fig. 7):

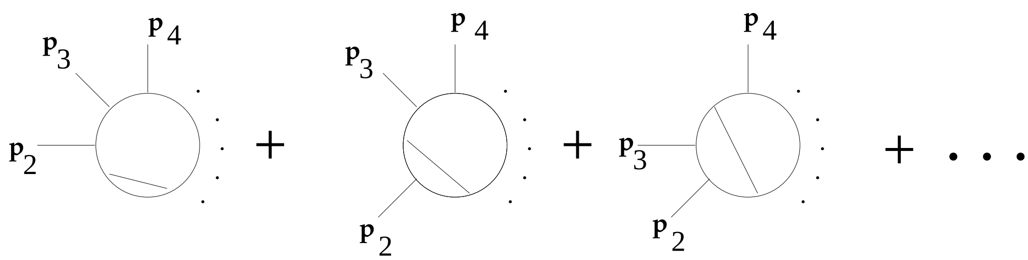


Figure 7: Sum of two – loop diagrams with different topologies.

This is interesting, as we have at hand a single integral formula for a sum containing many diagrams of different topologies. We may think of it as a remnant of the "less fragmented" nature of string perturbation theory mentioned before (fig. 3). Moreover, it calls certain well-known cancellations to mind which happen in gauge theory due to the fact that the Feynman diagram calculation splits a gauge invariant amplitude into gauge non-invariant pieces. For instance, to obtain the 3-loop β – function coefficient for quenched (single spinor - loop) QED, one needs to calculate the sum of diagrams shown in fig. 8.

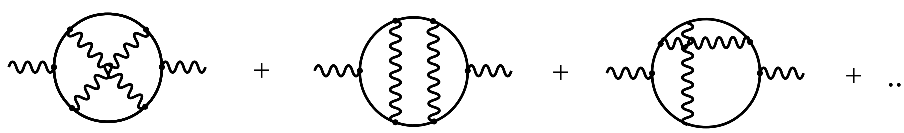


Figure 8: Sum of diagrams contributing to the 3-loop QED β – function.

Performing this calculation in, say, dimensional regularization, one finds that

1. All poles of order higher than $\frac{1}{\varepsilon}$ cancel.
2. Individual diagrams give contributions to the β – function proportional to $\zeta(3)$ which cancel in the sum, leaving a rational coefficient.

The first property is known to be a consequence of gauge invariance, and to hold true to all orders of perturbation theory [67]. The second one, i.e. the absence of irrational numbers in the quenched QED β – function, has been explicitly verified to four-loop order in spinor QED [68,69], and to three-loop order in scalar QED [70]. Recently arguments from knot theory have

[3] This fact was not obvious in the original string-theoretic derivation of (1.18), since before the infinite string tension limit the requirement of conformal invariance forces the external states to be on-shell.

been given which link both properties [70], indicating that this property should hold to all orders, too. As every practician in quantum field theory knows, similar extensive cancellations abound in calculations in gauge theory.

It seems therefore very natural to apply the Bern-Kosower formalism to this type of calculation. However, in its original version the Bern-Kosower formalism was confined to tree–level and one–loop amplitudes. The extension of this formalism beyond one-loop is obviously desirable, and has already been attempted along quite different lines:

In the original approach of Bern and Kosower, going beyond one loop would imply finding the particle theory limits of higher genus string amplitudes, a formidable task considering the complicated structure of moduli space for genus higher than one. While a suitable representation of the N - gluon amplitude at arbitrary genus was already given in [71], and substantial progress was achieved in the analysis of the infinite string tension limit [72,73,74,75,76,77,78], so far this line of work has not yet led to the formulation of multiloop Bern-Kosower type rules.

Another, and in some sense opposite route has been taken by Lam [79], who sets out with the usual Feynman parameter integral representation of multiloop diagrams, and uses the electric circuit analog [80,81] to transform those into the Koba-Nielsen type representation which one would expect from a string-type calculation. Yet another approach has been followed by McKeon [82], who proposes to perform multiloop calculations by writing a separate worldline path integral for every internal propagator of a diagram. A Hamiltonian approach was considered in [83].

In principle one could, of course, also construct Bern-Kosower type multiloop formulae using the explicit sewing procedure indicated above. However, we will describe another multiloop formalism here, proposed by M.G. Schmidt and the author [84,85], which is based on a more efficient way of inserting propagators into one-loop amplitudes. This approach is a direct generalization of Strassler's one-loop formalism, and preserves its main properties. Its distinguishing features are the following:

1. We will generalize eq.(1.16) for the one-loop Green's function to the construction of Green's functions defined on multiloop graphs.

2. The superfield formalism for the fermion loop will carry over to the multiloop level.

3. All field theory vertices will be represented by worldline quantities.

Though not explicitly referring to string theory any more, the resulting formalism may still be called "string-inspired" in the sense that it has a natural interpretation in terms of a one - dimensional field theory defined on graphs. As one would expect from a generalization of the Bern-Kosower method, it allows one to derive well-organized parameter integral representations for dimensionally regularized off-shell amplitudes, without the need for computing momentum integrals or Dirac traces.

At the multiloop level, this formalism has been worked out comprehensively for scalar field theories [84,86,87,88,89,90] as well as for scalar and spinor QED [85,91,92,93]. In those models, in principle it applies to the calculation of arbitrary off-shell amplitudes involving only spin 0 and spin 1 scattering states, or of the corresponding effective Lagrangians. More recently along these lines preliminary results have been obtained also for Yang-Mills Theory at the two-loop level [94,95].

Our applications center around the photon S-matrix in quantum electrodynamics as our main paradigm. Here the formalism has been developed to a point where it shows some distinct advantages over the more standard methods, in particular for problems involving constant external fields. At the one-loop level, we present also a number of calculations involving Yukawa and axial couplings, as well as non-abelian gauge fields.

The material is organized as follows. In chapter 2 we state the Bern-Kosower rules for the case of gluon scattering, and shortly sketch their derivation from the open bosonic string.

In chapter 3 we give derivations for the most basic worldline Lagrangians used in this work, describing the coupling of particles with spin 0, $\frac{1}{2}$, and 1 to external gauge fields. The starting point in this derivation is always the proper-time representation of the one-loop effective action in terms of the one-loop functional determinant. We discuss in particular detail the path integral representation of spin – 1 particles [64,92], since here the application of the string – inspired technique requires a non-standard approach.

The principle of how to calculate such path integrals within the "string-inspired formalism" is then explained in chapter 4. The advantages of the technique compared to the standard Feynman diagram technique are then demonstrated using the example of the QED and QCD vacuum polarizations, and QCD gluon – gluon scattering. We investigate the systematics of the Bern-Kosower partial integration procedure for the general N - photon / N gluon amplitudes, and determine the structure of the resulting integrand. We also clarify the relation of the worldline parameter integrals to the ones arising in standard Feynman parameter calculations of the same amplitudes.

Chapter 5 is devoted to the treatment of QED amplitudes in a constant electromagnetic background field. This case is given special attention since it provides a particularly natural application of the string-inspired technique [96,97,98,99,100,92].

In chapter 6 we consider more general field theories. While worldline path integral representations have been known and investigated for decades for the case of spin – 0 and spin – $\frac{1}{2}$ particles minimally coupled to gauge and gravitational backgrounds, worldline Lagrangians describing the coupling of a Dirac fermion to a general background consisting of a scalar, pseudoscalar, vector, axialvector and antisymmetric tensor field were obtained only recently [101,102,103,104,105]. In the present review we restrict ourselves to two special cases which admit particularly elegant formulations, namely the scalar – pseudoscalar and the vector – axialvector amplitudes.

Chapter 7 deals with the application of the formalism to the calculation of effective actions in the higher derivative or heat-kernel expansion [96,106,107]. While our discussion concentrates on the gauge theory case, we also shortly discuss some mathematical subtleties which arise in the generalization of the formalism to curved backgrounds, and which were clarified only very recently.

In chapter 8 we generalize the worldline formalism to the calculation of multiloop amplitudes in scalar self-interacting field theories. This generalization is based on the concept of Green's functions defined on graphs, and follows the string analogy closely. We derive explicit expressions for those Green's functions for a large class of graphs, the so-called "Hamiltonian graphs", and verify for some simple two-loop examples that their application reproduces the same amplitudes as the corresponding Feynman diagram calculations.

This formalism is then generalized to quantum electrodynamics in chapter 9, where in principle

it applies to the calculation of the whole photon S-matrix. At the two-loop level, we present a detailed recalculation [92,108] of the Euler-Heisenberg Lagrangians in scalar and spinor QED, including the β – function coefficients. An interesting cancellation which occurs in the spinor QED case is explained by an analysis of the renormalization procedure.

In the conclusions we give a short overview over the present range of applicability of the worldline technique, and point out some possible future directions.

There are several technical appendices. Appendix A contains a summary of the conventions used in the present work [4], including the rules for continuation from Euclidean to Minkowski space. In appendix B we give detailed calculations of the various worldline propagators which are used in the main text. Appendix C contains a more detailed discussion of the partial integration procedure introduced in section 4.8, and a summary of the resulting worldline integrands up to the six-point case. The results are used in appendix D for a simple proof of the basic fermionic replacement rule (2.15). In appendix E we explain a technique for the calculation of four-point massless on-shell tensor parameter integrals, following [109]. Finally, appendix F contains a collection of formulas which we have found useful, or at least amusing.

[4] Those differ in some points from previous work by this author.

2. The Bern – Kosower Formalism

This chapter is devoted to a statement of the Bern-Kosower rules, and to a short account of their derivation from string theory. We follow not the original derivation from the heterotic string [19,20,21] but the simpler one using the open bosonic string, as given in [25,66].

2.1. The Infinite String Tension Limit

For the calculation of the one-loop N - gluon amplitude for the open bosonic string, one inserts N copies of the gluon vertex operator eq.(1.4) on the boundary of the annulus, fig. 2. Then one has to compute the Polyakov path integral over the space of all embeddings of the annulus into spacetime, and the integral over moduli space. In the case of the annulus there is only one modular parameter [5],

$$\tau = -\frac{1}{2}\ln(q) \tag{2.1}$$

where q can be interpreted as the square of the ratio of the two radii defining the annulus. Since the two-dimensional worldsheet theory is free, the path integral can be computed by a repeated application of Wick's theorem, using the Green's function $G_B^{\rm ann}$ for the Laplacian on the annulus. If we assume all of the vertex operators to be on the same boundary, one has explicitly

$$\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle = g^{\mu\nu}G_B^{\rm ann}(\tau_1,\tau_2) = -g^{\mu\nu}\biggl[\ln|2\sinh(\tau_{12})| - \frac{\tau_{12}^2}{\tau} - 4q\sinh^2(\tau_{12})\biggr] + O(q^2) \tag{2.2}$$

where τ denotes the length of the boundary, τ_i the location of the i-th vertex operator along the boundary, $\tau_N = 0$, and $\tau_{ij} \equiv \tau_i - \tau_j$. One obtains a parameter integral (compare eq.(1.18))

$$\begin{aligned}\Gamma[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \quad &\sim \quad (\alpha')^{(\frac{N}{2}-2)}\int_0^\infty d\tau\tau^{-\frac{D}{2}}Z(\tau)\prod_{i=1}^{N-1}\int_0^\tau d\tau_i\,\theta(\tau_i-\tau_{i+1})\\ &\times\exp\biggl\{\sum_{i<j=1}^N[\alpha' G_{Bij}^{\rm ann}k_i\cdot k_j + \frac{1}{2}\sqrt{\alpha'}\dot G_{Bij}^{\rm ann}(k_i\cdot\varepsilon_j - k_j\cdot\varepsilon_i) - \frac{1}{4}\ddot G_{Bij}^{\rm ann}\varepsilon_i\cdot\varepsilon_j]\biggr\}\,|_{\rm multi-linear}\end{aligned} \tag{2.3}$$

Here we have omitted the color trace and some global factors. $Z(\tau)$ is essentially the string vacuum partition function, and given by

$$Z = q^{-1}\prod_{n=1}^\infty (1-q^n)^{-2} = q^{-1} + 2 + O(q) \tag{2.4}$$

The analysis of the infinite string tension limit $\alpha' \to 0$ can be simplified by first removing all second derivatives $\ddot G_{Bij}^{\rm ann}$ by suitable partial integrations in the variables τ_i. This is always possible [20], and will be discussed later on in the field theory context. The integrand then becomes homogeneous in α'. The possible boundary terms appearing in the integration by parts can be made to vanish by a suitable analytic continuation in the external momenta.

In the $\alpha' \to 0$ limit, first one has to extract massless poles in the S-matrix, which can appear for regions where $\tau_i \to \tau_j$. Those are of the form

$$\int d\tau_i \frac{1}{{\tau_{ij}}^{1+\alpha' k_i \cdot k_j}} \quad \stackrel{\alpha' \to 0}{\longrightarrow} \quad -\frac{1}{\alpha' k_i \cdot k_j} \tag{2.5}$$

and yield the so-called "tree" or "pinch" contributions. The limit itself is to be taken on the sum of the unpinched expression together with all pinch contributions. It is consumed by taking $\tau, |\tau_{ij}| \to \infty$, which corresponds to the ratio of radii approaching 1, and thus to the annulus being squeezed to a field theory loop. Analyzing G_B^{ann} and $\dot{G}_B^{\mathrm{ann}}$ in this limit, one finds that they can be replaced by

$$\begin{aligned}
\exp\Bigl[G_B^{\mathrm{ann}}(\tau_{12})\Bigr] \quad &\to \quad \text{const.} \times \exp\Bigl(\frac{\tau_{12}^2}{\tau} - |\tau_{12}|\Bigr) &(2.6)\\
\dot{G}_B^{\mathrm{ann}}(\tau_{12}) \quad &\to \quad -\mathrm{sign}(\tau_{12}) + 2\frac{\tau_{12}}{\tau} + 2\,\mathrm{sign}(\tau_{12})\Bigl(q\,\mathrm{e}^{2|\tau_{12}|} - \mathrm{e}^{-2|\tau_{12}|}\Bigr) &(2.7)
\end{aligned}$$

Since this limit corresponds to $q \to 0$, and the expansion of Z as a power series in q starts with a q^{-1}, there are again two types of contributions.

The first type arises by picking the next-to-leading constant term in Z. Then only the leading order terms from the integrand can survive the limit, so that $\dot{G}_B^{\mathrm{ann}}$ is further truncated to the first two terms of eq.(2.7).

The second type is obtained by combining the leading order term from Z with a next-to-leading term in the integrand. Then subleading terms in the $\dot{G}_{Bij}^{\mathrm{ann}}$ can potentially contribute, however it turns out that a too strong suppression of the integrand for $q \to 0$ can be avoided only for those terms in the integrand which contain a closed cycle of $\dot{G}_{Bij}^{\mathrm{ann}}$'s, i.e. a factor

$$\dot{G}_{Bi_1i_2}^{\mathrm{ann}} \dot{G}_{Bi_2i_3}^{\mathrm{ann}} \cdots \dot{G}_{Bi_ni_1}^{\mathrm{ann}} \tag{2.8}$$

Even then, the cycle can only survive the limit if the indices follow the ordering of the external legs. Of course it is also possible to combine the leading order terms from both Z and the integrand. This produces terms which diverge in the limit. Those are discarded, since they can be identified as being due to an unphysical tachyonic scalar circulating in the loop.

In this way one arrives at the Bern-Kosower rules for the gluon loop as given in [21,25]. In the following we will give those rules in a slightly different version, which is more in line with the worldline path integral approach with respect to the treatment of the color algebra. The original string-based approach naturally yields the gluon amplitudes in color-decomposed form, i.e. with the group theory factors expressed in terms of the fundamental instead of the adjoint representation [110,111]. If in the above derivation the gluon vertex operators are taken to be in the fundamental representation, as it was done in [21,25], then the resulting field theory amplitude represents the so-called "leading color partial amplitude", where leading refers to the large N_c limit of $SU(N_c)$ gauge theory. (There is an extra overall factor of N_c in this approach, coming from a second index in the fundamental representation, which is untouched by all the gluon vertex operators, leading to $\mathrm{tr}(1) = N_c$.) The missing subleading amplitudes would be obtained by the inclusion of string theory amplitudes with vertex operator insertions on both boundaries of the annulus, although it turns out that they can also be constructed directly as sums of permutations of the leading color amplitude [112,113].

The form of the spin 1 rules which we give here instead does not use color decomposition; in eqs.(2.18),(2.19) below, as well as in the remainder of this review, a color matrix T^a always refers to the adjoint representation in the gluon loop case. The equivalence of these rules to the original version of [21,25] follows from recent work on the color decomposition [113].

The analogous rules for the spin $\frac{1}{2}$ loop can be derived by repeating the same analysis for the open superstring. Remarkably, this leads to a rule which allows one to infer all contributions from worldsheet fermions to the final integrand from the purely bosonic terms. This "replacement rule" will play a prominent role in many of our applications. Its validity can be shown to be a consequence of worldsheet supersymmetry [66]. For the spin 0 loop, one finds only the first type of contributions above.

2.2. The Bern-Kosower Rules for Gluon Scattering

For the statement of the rules, we rewrite them in the conventions used throughout the remainder of this work. In particular, we work in the Euclidean.

The Bern-Kosower rules give a prescription for the construction of integral representations for one-loop photon or gluon scattering amplitudes in non-abelian gauge theory [21]. We write them down here for the case of a non-abelian gauge theory with massless scalars and fermions. To obtain the one-loop on-shell amplitude for the scattering of N gluons [5], with momenta k_i and polarization vectors ε_i, the following steps have to be taken:

step 1

Consider the following kinematic expression:

$$K = \int \prod_{i=1}^{N} du_i \prod_{i<j} \exp\Bigl[G_{Bij} k_i \cdot k_j + i\dot{G}_{Bij}(k_i \cdot \varepsilon_j - k_j \cdot \varepsilon_i) + \ddot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j\Bigr] \Big|_{\mathrm{multi-linear}} \tag{2.9}$$

where "multi-linear" means that only terms linear in each of the $\varepsilon_1, \ldots, \varepsilon_N$ are to be kept.

step 2

An on-shell gluon has only two physical polarizations, denoted by "+" and "–". Consider one helicity amplitude at a time, and denote, for example, by A(+,+,–,...,–) the amplitude for the process where the first two gluons have the same helicity, and all remaining ones the opposite one. Each helicity amplitude is separately gauge invariant, i. e. insensitive to a redefinition of any of the polarization vectors by a transformation

$$\varepsilon_i^{\pm\mu} \to \varepsilon_i^{\pm\mu} + \lambda k_i^{\mu} \tag{2.10}$$

This freedom can be used to choose – for given external momenta $\{k_1, \ldots, k_N\}$ – a set of polarization vectors which makes a maximal number of the invariants $k_i \cdot \varepsilon_j$ and $\varepsilon_i \cdot \varepsilon_j$ vanish. A systematic way of finding such a set of polarization vectors for a given choice of helicities is

[5] By "gluon" we denote any non-abelian gauge boson.

provided by the *Spinor Helicity Method* (see, e.g., [66,114]). At the end, all surviving invariants are rewritten as functions of the external momenta alone.

step 3

Expand out the kinematic expression, and perform integrations by parts, till all double derivatives of G_B are removed (ignore boundary terms). We have now an expression

$$\int \prod_{i=1}^{N} du_i K_{\text{red}} \prod_{i<j} \exp[G_{Bij} k_i \cdot k_j] \tag{2.11}$$

where K_{red}, the "reduced kinematic factor", is a sum of terms that are products of $\dot{G}_{Bij}$'s, and of dot products $k_i \cdot k_j, k_i \cdot \varepsilon_j, \varepsilon_i \cdot \varepsilon_j$.

step 4

Draw all possible labelled ϕ^3 1-loop diagrams D_i with N external legs,

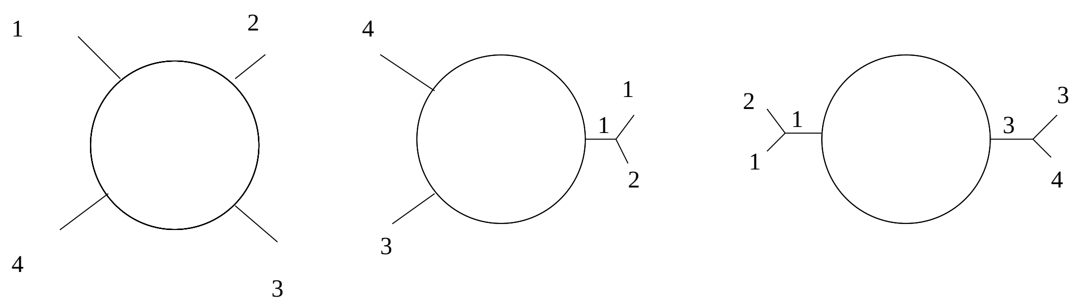


Figure 9: Diagrams in ϕ^3 theory.

but excluding tadpoles, and diagrams where the loop is isolated on an external leg:

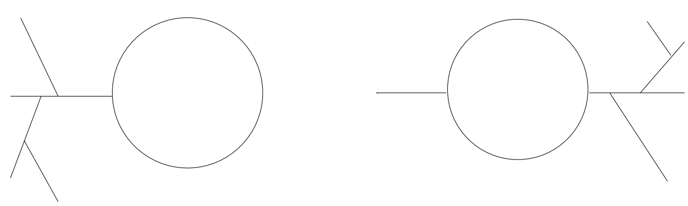


Figure 10: Diagrams to be omitted.

The labels follow the cyclic ordering of the trace. One also attaches a label to every internal

line in the tree part of a diagram; for definiteness, this is taken to be the smallest one of the labels of the two lines into which the line splits to the outward.

Every diagram D_i contributes a parameter integral

$$D_i = \Gamma(m - \frac{D}{2}) \int_0^1 du_{i_1} \int_0^{u_{i_1}} du_{i_2} \cdots \int_0^{u_{i_{m-2}}} du_{i_{m-1}} \frac{P_i(u_{i_1}, \ldots, u_{i_m})}{[-\sum_{r<s}^m G_{Bi_ri_s} K_{i_r} \cdot K_{i_s}]^{m-\frac{D}{2}}} \tag{2.12}$$

Here D is the space-time dimension, m is the number of legs directly attached to the loop, and

$$G_{Bi_ri_s} = G_B(u_{i_r}, u_{i_s}) = \mid u_{i_r} - u_{i_s} \mid - (u_{i_r} - u_{i_s})^2 = (u_{i_r} - u_{i_s}) - (u_{i_r} - u_{i_s})^2 \tag{2.13}$$

($u_{i_m} = 0$). K_{i_r} denotes the sum of the external momenta flowing into the tree which enters the loop at the point carrying the label i_r. P_i is a polynomial function of the loop parameters, and of the external momenta and polarization vectors; it will be determined in steps 5 and 6.

step 5: **tree replacement rules**

Remove all trees, working from the outside of the diagram toward the loop. If the diagram contains a vertex as shown in fig. (11)

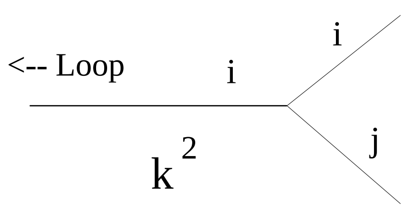


Figure 11: Removal of trees.

keep only those terms in K_{red} which contain exactly one $\dot{G}_{Bij}$. In those, replace $\dot{G}_{Bij}$, with $i < j$, by $\frac{2}{k^2}$, and replace all remaining $\dot{G}_{Bjr}$ by $\dot{G}_{Bir}$. Repeat this procedure, till only the naked loop is left.

step 6: **loop replacement rules**

It is only at this stage that one has to distinguish between the scalar, the fermion, and the gluon loop.

Scalar loop: Simply write out K_{red} (i. e. what became of K_{red} in step 5) in terms of the integration variables, by substituting

$$\dot{G}_{Bij} \rightarrow \text{sign}(u_i - u_j) - 2(u_i - u_j) \tag{2.14}$$

Multiply by an overall factor of 2 if the scalar is complex.

Spinor loop: Replace *simultaneously* every closed cycle $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ki_1}$ appearing in $K_{\rm red}$ (which may or may not follow the ordering of the external legs) by

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ki_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ki_1} \tag{2.15}$$

An expression is considered a cycle already if it can be put into cycle form using the antisymmetry of $\dot G_B$ (e.g. $\dot G_{Bij}\dot G_{Bij} = -\dot G_{Bij}\dot G_{Bji}$).

Then replace all $\dot G_B$'s as in the scalar case, and all G_F's by

$$G_{Fij} \to {\rm sign}(u_i - u_j) \tag{2.16}$$

Multiply by an overall factor of

–4 for a Dirac fermion

–2 for a Weyl fermion

Gluon loop: In this case, there are two types of contributions, which have to be summed:

type 1: Replace $\dot G_{Bij}$ as above.

type 2: For every closed cycle of $\dot G_{Bij}$'s appearing in a term, *with the ordering of the indices following the ordering of the external legs*, write down *one* additional contribution, obtained in the following way: replace

$$\begin{aligned} \dot G_{Bi_1i_2}\dot G_{Bi_2i_1} &\to 4 \\ \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ki_1} &\to 2^{k-1} \quad (k>2) \end{aligned} \tag{2.17}$$

and all remaining $\dot G_B$'s – even those belonging to other cycles – as in the scalar case.

Multiply by a factor of 2 for both types.

The expression obtained from $K_{\rm red}$ in this way is the polynomial P_i above.

step 7

Perform the parameter integrations. Techniques for their calculation may be found in [109,115,116]. The four-point case is treated in appendix E, following [109].

step 8

Finally the amplitude is given by a sum over all diagrams, with an overall normalization factor:

$$\Gamma^{a_1\cdots a_N}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = (-ig)^N {\rm tr}(T^{a_1}\cdots T^{a_N})\frac{(4\pi\mu^2)^{-\frac{\epsilon}{2}}}{32\pi^2}\sum_{\rm diagrams} D_i \tag{2.18}$$

Here $\epsilon = D-4$, and μ is the usual unit of mass appearing in the dimensional continuation. T^{a_i} is a color matrix in the representation of the loop particle. We have specialized to a

specific version of dimensional regularization, the so-called four-dimensional helicity scheme [21]. This is only the partial amplitude corresponding to the considered fixed ordering of the external states. To obtain the complete amplitude one must still sum over all possible non-cyclic permutations of the states, so that

$$\Gamma(\{a_i, k_i, \varepsilon_i\}) = \sum_{\pi \in S_N / Z_N} \Gamma^{a_{\pi(1)} \cdots a_{\pi(N)}}[k_{\pi(1)}, \varepsilon_{\pi(1)}; \ldots; k_{\pi(N)}, \varepsilon_{\pi(N)}] \tag{2.19}$$

In chapter four we will explicitly apply these rules to the four-gluon case.

See [27,28] for the analogous rules for graviton scattering in quantum gravity.

3. Worldline Path Integral Representations for Effective Actions

In this chapter, we derive worldline path integral representations for a number of one-loop effective actions involving some of the most basic interactions in quantum field theory. Those derivations are based on the fact that one-loop effective actions can generally be expressed in terms of the determinant of the kinetic operator in field theory. Using the $ln(det) = tr(ln)$ – formula and the Schwinger proper-time representation, one obtains an integral over the space of all closed trajectories of a quantum mechanical particle moving in spacetime.

Generally, to every such closed loop path integral one finds associated a similar open-ended path integral, representing the field theory propagator of the loop particle in the background field. The propagator path integral has the same worldline Lagrangian, possibly with some boundary terms added. It is to be performed over the space of trajectories connecting two fixed points in spacetime, with appropriate boundary conditions.

In the present review we will concentrate on the effective action, or closed loop case, simply because almost all explicit calculations which have been done so far pertain to this case. By this we do not mean to imply that propagator path integrals may not play an important role in future extensions of this formalism.

Our derivations are mostly formal. Only in the spin – 1 loop case will we touch upon the subtle issues connected with the existence of different discretization prescriptions etc. Those have recently been investigated in much detail for the case of curved backgrounds, a subject which will be shortly discussed in section 7.2.

3.1. Scalar Field Theory

Let us begin with the simplest case of a real massive scalar field ϕ with a self-interaction potential $U(\phi)$. According to standard quantum field theory (see, e.g., [117]) the Euclidean one-loop effective action for this field theory can be written as [6]

$$\Gamma[\phi] = -\frac{1}{2}\mathrm{Tr}\ln\Big[\frac{-\Box + m^2 + U''(\phi)}{-\Box + m^2}\Big] \tag{3.1}$$

We use the formula

$$-\,\mathrm{Tr}\,\ln\Big(\frac{A}{B}\Big) = \int_0^\infty \frac{dT}{T}\mathrm{Tr}\left(\mathrm{e}^{-AT} - \mathrm{e}^{-BT}\right) \tag{3.2}$$

valid for positive definite operators A, B, delete the irrelevant ϕ – independent term, and perform the functional trace in x – space. This gives

$$\Gamma[\phi] = \frac{1}{2}\int_0^\infty \frac{dT}{T}\int d^Dx\,\langle x \mid \exp\Big\{-T\Big[-\Box + m^2 + U''(\phi(x))\Big]\Big\} \mid x\rangle \tag{3.3}$$

Now compare this with Feynman's path integral formula for the evolution operator in non-relativistic quantum mechanics. For a particle with mass $\tilde{m}$ moving in a time-independent

[6] We work with relativistic quantum field theory conventions, $\hbar = c = 1$. Functional traces are denoted by Tr, finite dimensional traces by tr.

potential $\tilde{V}(x)$ this formula reads (see, e.g., [118,119]),

$$\langle x'' \mid \mathrm{e}^{-i(t''-t')H} \mid x' \rangle = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t)\, \mathrm{e}^{i\int_{t'}^{t''} dt \left[\frac{\tilde{m}}{2}\dot{x}^2 - \tilde{V}(x)\right]} \tag{3.4}$$

We can therefore interpret our kinetic operator above as the Hamilton operator H for a fictitious particle moving in D dimensions,

$$H = \frac{p^2}{2\tilde{m}} + \tilde{V}(x) \tag{3.5}$$

by identifying

$$\begin{aligned} \tilde{V}(x) &= m^2 + U''(\phi(x)) \\ \tilde{m} &= \frac{1}{2} \\ i(t''-t') &= T \end{aligned} \tag{3.6}$$

Without retracing the usual path integral discretization procedure [118,119] we can thus immediately write

$$\langle x \mid \exp\Bigl\{-T\Bigl[-\Box + m^2 + U''(\phi(x))\Bigr]\Bigr\} \mid x \rangle = \int_{x(0)=x}^{x(T)=x} \mathcal{D}x(\tau) \mathrm{e}^{-\int_0^T d\tau \left[\frac{1}{4}\dot{x}^2 + m^2 + U''(\phi(x(\tau)))\right]} \tag{3.7}$$

$(\tau = it)$. Taking into account that

$$\int d^D x \int_{x(0)=x(T)=x} \mathcal{D}x(\tau) = \int_{x(0)=x(T)} \mathcal{D}x(\tau) \tag{3.8}$$

we obtain the desired path integral representation for the effective action,

$$\Gamma[\phi] = \frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + U''(\phi(x(\tau)))\right)} \tag{3.9}$$

In a completely analogous way one derives the path integral representation for the scalar propagator in the background field ϕ,

$$\begin{aligned} \langle x'' \mid \Bigl[-\Box + m^2 + U''(\phi(x))\Bigr]^{-1} \mid x' \rangle &= \int_0^\infty dT\, \langle x'' \mid \exp\Bigl\{-T\Bigl[-\Box + m^2 + U''(\phi(x))\Bigr]\Bigr\} \mid x' \rangle \\ &= \int_0^\infty dT\, \mathrm{e}^{-m^2 T} \int_{x(0)=x'}^{x(T)=x''} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + U''(\phi(x(\tau)))\right)} \end{aligned} \tag{3.10}$$

3.2. Scalar Quantum Electrodynamics

The path integral for a massive (complex) scalar field minimally coupled to a background Maxwell field can also be found simply by recurring to quantum mechanics. The field theory kinetic operator now reads

$$(\partial + ieA)^2 - m^2 \tag{3.11}$$

with a fictitious Hamiltonian

$$H = \frac{(p + eA)^2}{2\tilde{m}} + m^2 \tag{3.12}$$

This translates into

$$\begin{aligned}\Gamma_{\rm scal}[A] &= -\frac{1}{2}\mathrm{Tr}\ln\Big[\frac{-(\partial + ieA)^2 + m^2}{-\Box + m^2}\Big] \\ &= \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ie\,\dot{x}\cdot A(x(\tau))\right)}\end{aligned} \tag{3.13}$$

and analogously for the propagator. Note that the global factor of $\frac{1}{2}$ has disappeared, since in taking the trace we have to take the double number of degrees of freedom of the complex scalar into account.

3.3. Spinor Quantum Electrodynamics

For the spin $\frac{1}{2}$ - particle various worldline path integral representations can be found in the literature. The basic choice is between bosonic [29,30] and Grassmann [31,32,33,34,35,36,37,38,39,40] representations. The latter can be derived using either coherent state methods [120,121,102,104] or the "Weyl symbol" method [34,122].

Our following treatment of the spin $\frac{1}{2}$ – case uses the coherent state formalism. We would like to find a path integral representation for the Euclidean effective action [7]

$$\Gamma_{\rm spin}[A] = \ln \mathrm{Det}[\not{p} + e\not{A} - im] \tag{3.14}$$

This time we will not be able to just take over results from quantum mechanics; we have to construct our path integral by brute force.

Let us start with the well-known observation that we can rewrite

$$(\not{p} + e\not{A})^2 = -(\partial_\mu + ieA_\mu)^2 - \frac{i}{2}\, e\sigma^{\mu\nu}F_{\mu\nu} \tag{3.15}$$

($\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$). Using the usual argument that

$$\mathrm{Det}\Big[(\not{p} + e\not{A}) - im\Big] = \mathrm{Det}\Big[(\not{p} + e\not{A}) + im\Big] = \mathrm{Det}^{1/2}\Big[(\not{p} + e\not{A})^2 + m^2\Big] \tag{3.16}$$

[7] Our Euclidean Dirac matrix conventions are $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}, \gamma_\mu^\dagger = \gamma_\mu, \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$.

we can then write the effective action in the following form,

$$\Gamma_{\rm spin}[A] = -\frac{1}{2}\,{\rm Tr}\,\int_0^\infty \frac{dT}{T}\,\exp\Bigl\{-T\left[-(\partial+ieA)^2 - \frac{i}{2}e\,\sigma^{\mu\nu}F_{\mu\nu} + m^2\right]\Bigr\} \tag{3.17}$$

Up to the global sign, this is formally identical with the effective action for a scalar loop in a background containing, besides the gauge field A, a potential term

$$V \equiv -\frac{i}{2}e\,\sigma^{\mu\nu}F_{\mu\nu} \tag{3.18}$$

We will now use the formalism developed in [120] to transform this functional trace into a quantum mechanical path integral. Our treatment closely parallels the one in [102,104], except that they work in six dimensions. Define matrices a_r^+ and a_r^-, $r = 1,2$, by

$$a_1^\pm = \frac{1}{2}(\gamma_1 \pm i\gamma_3), \quad a_2^\pm = \frac{1}{2}(\gamma_2 \pm i\gamma_4) \tag{3.19}$$

Those satisfy Fermi-Dirac anticommutation rules

$$\{a_r^+, a_s^-\} = \delta_{rs}, \quad \{a_r^+, a_s^+\} = \{a_r^-, a_s^-\} = 0 \tag{3.20}$$

Thus we can use a_r^+ and a_r^- as creation and annihilation operators for a Hilbert space with a vacuum defined by

$$a_r^-|0\rangle = \langle 0 \mid a_r^+ = 0 \tag{3.21}$$

Next we introduce Grassmann variables η_r and $\bar\eta_r$, $r = 1,2$, which anticommute with one another and with the operators $a_r^\pm$, and commute with the vacuum $\mid 0\rangle$. The coherent states are then defined as

$$\begin{aligned} \langle\eta \mid &\equiv i\langle 0 \mid (\eta_1 - a_1^-)(\eta_2 - a_2^-) \qquad & \mid\eta\rangle &\equiv \exp(-\eta_1 a_1^+ - \eta_2 a_2^+) \mid 0\rangle \\ \langle\bar\eta \mid &\equiv \langle 0 \mid \exp(-a_1^-\bar\eta_1 - a_2^-\bar\eta_2) \qquad & \mid\bar\eta\rangle &\equiv i(\bar\eta_1 - a_1^+)(\bar\eta_2 - a_2^+) \mid 0\rangle \end{aligned} \tag{3.22}$$

It is easily verified that those satisfy the defining equations for coherent states,

$$\begin{aligned} \langle\eta \mid a_r^- &= \langle\eta \mid \eta_r \qquad & a_r^- \mid \eta\rangle &= \eta_r \mid \eta\rangle \qquad & \langle\eta \mid \bar\eta\rangle &= \exp(\eta_1\bar\eta_1 + \eta_2\bar\eta_2) \\ \langle\bar\eta \mid a_r^+ &= \langle\bar\eta \mid \bar\eta_r \qquad & a_r^+ \mid \bar\eta\rangle &= \bar\eta_r \mid \bar\eta\rangle \qquad & \langle\bar\eta \mid \eta\rangle &= \exp(\bar\eta_1\eta_1 + \bar\eta_2\eta_2) \end{aligned} \tag{3.23}$$

Also one introduces the corresponding Grassmann integrals, defined by

$$\int \eta_i d\eta_i = \int \bar\eta_i d\bar\eta_i = i \tag{3.24}$$

The $d\eta_r, d\bar\eta_r$ commute with one another and with the vacuum, and anticommute with all Grassmann variables and the $a_r^\pm$. This leads to the completeness relations

$$\mathbb{1} = i\int \mid\eta\rangle\langle\eta \mid d^2\eta = -i\int d^2\bar\eta \mid \bar\eta\rangle\langle\bar\eta \mid \tag{3.25}$$

($d^2\eta = d\eta_2 d\eta_1,\ d^2\bar\eta = d\bar\eta_1 d\bar\eta_2$), and to the following representation for a trace in the Fock space generated by the $a_r^\pm$,

$$\mathrm{Tr}\,(U) = i\int d^2\eta\,\langle -\eta \mid U \mid \eta\rangle \tag{3.26}$$

We can now apply these fermionic coherent states together with the usual complete sets of coordinate states to rewrite the functional trace in (3.17) in the following way,

$$\begin{aligned}\mathrm{Tr}\,\mathrm{e}^{-T\Sigma} &= i\int d^4x\int d^2\eta\,\langle x,-\eta \mid \mathrm{e}^{-T\Sigma} \mid x,\eta\rangle \\ &= i^N\int\prod_{i=1}^{N}\Bigl(d^4x^i d^2\eta^i\langle x^i,\eta^i \mid \mathrm{e}^{-\frac{T}{N}\Sigma} \mid x^{i+1},\eta^{i+1}\rangle\Bigr)\end{aligned} \tag{3.27}$$

where $\Sigma = -(\partial + ieA)^2 + V$. The boundary conditions on the x and η integrations are $(x^{N+1},\eta^{N+1}) = (x^1,-\eta^1)$. For the evaluation of this matrix element it will be useful to look first at the matrix elements of products of Dirac matrices. For the product of two γ's one finds

$$\langle\eta^i \mid \gamma_\mu\gamma_\nu \mid \eta^{i+1}\rangle = -i\int d^2\bar\eta^{i,i+1}\langle\eta^i \mid \bar\eta^{i,i+1}\rangle\langle\bar\eta^{i,i+1} \mid \eta^{i+1}\rangle 2\,{}^i\psi_\mu\psi_\nu^{i+1},\quad \mu\neq\nu \tag{3.28}$$

where

$$\begin{aligned}\psi_{1,2}^{i+1} &\equiv \frac{1}{\sqrt2}(\eta_{1,2}^{i+1} + \bar\eta_{1,2}^{i,i+1}), &\qquad \psi_{3,4}^{i+1} &\equiv \frac{i}{\sqrt2}(\eta_{1,2}^{i+1} - \bar\eta_{1,2}^{i,i+1}),\\ {}^i\psi_{1,2} &\equiv \frac{1}{\sqrt2}(\eta_{1,2}^{i} + \bar\eta_{1,2}^{i,i+1}), & {}^i\psi_{3,4} &\equiv \frac{i}{\sqrt2}(\eta_{1,2}^{i} - \bar\eta_{1,2}^{i,i+1})\end{aligned} \tag{3.29}$$

To verify this equation one rewrites the Dirac matrices in terms of the $a_r^\pm$ and then inserts a complete set of coherent states $\mid\bar\eta^{i,i+1}\rangle$ in between them.

With this information it is now easy to compute that

$$\begin{aligned}\langle x^i,\eta^i \mid \mathrm{e}^{-\frac{T}{N}\Sigma[p,A,\gamma_\mu\gamma_\nu]} \mid x^{i+1},\eta^{i+1}\rangle &= -\frac{i}{(2\pi)^4}\int d^4p^{i,i+1}d^2\bar\eta^{i,i+1}\mathrm{e}^{i(x^i-x^{i+1})p^{i,i+1}+(\eta^i-\eta^{i+1})_r\bar\eta_r^{i,i+1}}\\ &\quad\times\Bigl\{1-\frac{T}{N}\Sigma\Bigl[p^{i,i+1},A^{i,i+1},2\,{}^i\psi_\mu\psi_\nu^{i+1}\Bigr]+O\Bigl(\frac{T^2}{N^2}\Bigr)\Bigr\}\end{aligned} \tag{3.30}$$

Here the superscript $\phi^{i,i+1}$ on a field denotes the average of the corresponding fields with superscripts i and $i+1$. Inserting this result back into eq.(3.27) one obtains, after symmetrizing the positions of the Grassmann variables in the exponentials,

$$\begin{aligned}\mathrm{Tr}\,\mathrm{e}^{-T\Sigma} &= \frac{1}{(2\pi)^{4N}}\int\prod_{i=1}^{N}d^4x^i d^4p^{i,i+1}d^2\eta^i d^2\bar\eta^{i,i+1}\Bigl(1-\frac{T}{N}\Sigma_i+\mathrm{O}(\frac{T^2}{N^2})\Bigr)\\ &\quad\times\exp\Bigl\{\sum_{i=1}^{N}\Bigl[i(x^i-x^{i+1})p^{i,i+1}+\frac12(\eta_r^i-\eta_r^{i+1})\bar\eta_r^{i,i+1}-\frac12\eta_r^i(\bar\eta_r^{i-1,i}-\bar\eta_r^{i,i+1})\Bigr]\Bigr\}\end{aligned} \tag{3.31}$$

Introducing an interpolating proper-time τ such that $\tau_1 = T$, $\tau^{N+1} = 0$, and $\tau^i - \tau^{i+1} = \frac{T}{N}$, and taking the limit $N \to \infty$ in the usual naive way, we finally obtain the following path integral representation,

$$\mathrm{Tr}\,\mathrm{e}^{-T\Sigma} = \int \mathcal{D}p \int \mathcal{D}x \int_A \mathcal{D}\eta\mathcal{D}\bar{\eta} \exp\biggl\{\int_0^T d\tau\, \Bigl[i\dot{x}\cdot p + \frac{1}{2}\dot{\eta}_r\bar{\eta}_r - \frac{1}{2}\eta_r\dot{\bar{\eta}}_r - \Sigma[p, A, 2\psi_\mu\psi_\nu]\Bigr]\biggr\} \tag{3.32}$$

The "A" denotes the antiperiodic boundary conditions which we have for $\eta, \bar{\eta}$.

The continuum limits of eqs.(3.29) are

$$\psi_{1,2}(\tau) = \frac{1}{\sqrt{2}}(\eta_{1,2}(\tau) + \bar{\eta}_{1,2}(\tau)), \qquad \psi_{3,4}(\tau) = \frac{i}{\sqrt{2}}(\eta_{1,2}(\tau) - \bar{\eta}_{1,2}(\tau)) \tag{3.33}$$

This suggests a change of variables from $\eta, \bar{\eta}$ to ψ, which we complete by rewriting the fermionic kinetic term,

$$\frac{1}{2}\dot{\eta}_r\bar{\eta}_r - \frac{1}{2}\eta_r\dot{\bar{\eta}}_r = -\frac{1}{2}\psi^\mu\dot{\psi}_\mu \tag{3.34}$$

The boundary conditions are now $(x(T), \psi(T)) = (x(0), -\psi(0))$. Finally, we note that the momentum path integral is Gaussian, and perform it by a naive completion of the square (for a less unscrupulous treatment of this point see again [102,104], as well as for the various normalization factors involved). This brings us to our following final result [8]

$$\Gamma_{\mathrm{spin}}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int_P \mathcal{D}x \int_A \mathcal{D}\psi\, \mathrm{e}^{-\int_0^T d\tau\, L_{\mathrm{spin}}} \tag{3.35}$$

$$L_{\mathrm{spin}} = \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi_\mu\dot{\psi}^\mu + ie\dot{x}^\mu A_\mu - ie\psi^\mu F_{\mu\nu}\psi^\nu \tag{3.36}$$

which we already quoted in the introduction, eq.(1.9). Although in the present review we will be exclusively concerned with four-dimensional field theories, it should be mentioned that the obtained path integral representation is valid for all even spacetime dimensions. Note also that only the even subspace of the Clifford algebra came into play in the above. This is different in the case of an open fermion line, and is the reason why the corresponding path integral representation for the electron propagator in a background field is significantly more complicated [38,123,121,39,40,92].

In the introduction it was also mentioned that the worldline Lagrangian (3.36) has a global supersymmetry, (1.10). One consequence of this is that we can make use of a one-dimensional superfield formalism. Introducing

$$X^\mu = x^\mu + \sqrt{2}\,\theta\psi^\mu \tag{3.37}$$

$$Y^\mu = X^\mu - x_0^\mu \tag{3.38}$$

$$D = \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial\tau} \tag{3.39}$$

$$\int d\theta\theta = 1 \tag{3.40}$$

[8] Our definition of the Euclidean effective action differs by a sign from the one used in [102,104].

we can combine the $x-$ and $\psi-$ path integrals into the following super path integral [63,124,125,85],

$$\Gamma_{\rm spin}[A] = -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int \mathcal{D}X\, \mathrm{e}^{-\int_0^T d\tau \int d\theta\, \left[-\frac{1}{4}X\cdot D^3X - ieDX\cdot A(X)\right]} \tag{3.41}$$

Written in this way, the spinor path integral becomes formally analogous to the scalar one, and can be considered as its "supersymmetrization". Note, however, that the supersymmetry is broken by the different periodicity conditions which we have for the coordinate and the Grassmann path integrals. For a constant Grassmann parameter η those are not respected by the supersymmetry transformations (1.10).

3.4. Non-Abelian Gauge Theory

3.4.1. Scalar Loop Contribution to the Gluon Effective Action

The simplest non-abelian generalization which one can consider is the contribution to the gluon scattering amplitude due to a scalar loop. Retracing the above derivation of the scalar path integral for photon scattering, one finds that the non-abelian nature of the background field leads to the following changes in eq.(1.8):

1. The trace now includes a global color trace.
2. The corresponding quantum mechanical Hamilton operators at different times need not commute any more, so that the exponential must be taken path-ordered.

We have thus

$$\Gamma_{\rm scal}[A] = \mathrm{tr}\int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int \mathcal{D}x(\tau)\, \mathcal{P} e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + ig\dot{x}\cdot A(x(\tau))\right)} \tag{3.42}$$

where now $A_\mu = A_\mu^a T^a$. $\mathcal{P}$ denotes the path ordering operator, and tr the color trace.

3.4.2. Spinor Loop Contribution to the Gluon Effective Action

In addition to these two changes, in the spinor loop case the $F_{\mu\nu}$ appearing in the worldline Lagrangian (3.36) must now be taken to be the full non-abelian field strength tensor, including the commutator term $[A_\mu, A_\nu]$ [126,127].

One may wonder how this commutator term is to be accommodated in the superfield formalism. As was shown in [125], a very convenient way of doing so is to introduce a super path ordering. The ordinary path ordering can be defined by

$$\mathcal{P}\prod_{i=1}^N \int_0^T d\tau_i \equiv N! \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{N-1}} d\tau_N = N! \int_0^T d\tau_1 \cdots \int_0^T d\tau_N \prod_{i=1}^{N-1} \theta(\tau_i - \tau_{i+1}) \tag{3.43}$$

The super path ordering is obtained from this simply by replacing the proper-time differences in the arguments of the θ – functions by super-differences,

$$\hat{\tau}_{ij} \equiv \tau_i - \tau_j + \theta_i\theta_j \tag{3.44}$$

so that

$$\hat{\mathcal{P}}\prod_{i=1}^{N}\int_0^T d\tau_i \int d\theta_i \equiv N! \int_0^T d\tau_1 \int d\theta_1 \cdots \int_0^T d\tau_N \int d\theta_N \prod_{i=1}^{N-1}\theta(\hat{\tau}_{i(i+1)}) \tag{3.45}$$

(here and in the following we use the convention that $\prod_{i=1}^{N} d\theta_i \equiv d\theta_1 d\theta_2 \cdots d\theta_N$). Then expanding the θ – functions one finds

$$\theta(\hat{\tau}_{i(i+1)}) = \theta(\tau_i - \tau_{i+1}) + \theta_i\theta_{i+1}\delta(\tau_i - \tau_{i+1}) \tag{3.46}$$

and the δ – function terms will generate precisely the commutator terms above. With this definition of $\hat{\mathcal{P}}$, we can thus generalize eq.(3.41) to the non-abelian case as

$$\Gamma_{\rm spin}[A] = -\frac{1}{2}{\rm tr}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int \mathcal{D}X \hat{\mathcal{P}}\, {\rm e}^{-\int_0^T d\tau \int d\theta \left[-\frac{1}{4}X\cdot D^3 X - igDX\cdot A(X)\right]} \tag{3.47}$$

This remarkable interplay between worldline supersymmetry and spacetime gauge symmetry has recently attracted some attention [128].

3.4.3. *Gluon Loop Contribution to the Gluon Effective Action*

We proceed to the much more delicate case of the gluon loop, i.e. we wish now to derive a path integral describing a spin-1 particle coupled to a spin-1 background. Here one would expect to run into difficulties. It is well-known how to construct free path integrals for particles of arbitrary spin (see, e.g., [129,130]). However, the quantization of those path integrals usually leads to inconsistencies as soon as one tries to couple a path integral with spin higher than $\frac{1}{2}$ to a spin-1 background. In [64] this problem had been circumvented by the introduction of auxiliary degrees of freedom, and we will follow a similar approach here [92].

We employ the background gauge fixing technique so that the effective action $\Gamma[A^a_\mu]$ becomes a gauge invariant functional of A^a_μ [131,132]. The gauge fixed classical action reads, in D dimensions,

$$S[a;A] = -\frac{1}{4}\int d^Dx F^a_{\mu\nu}(A+a)F^{a\mu\nu}(A+a) - \frac{1}{2\alpha}\int d^Dx \left(D^{ab\mu}[A]a^b_\mu\right)^2 \tag{3.48}$$

A priori, the background field A^a_μ is unrelated to the quantum field a^a_μ. The kinetic operator of the gauge boson fluctuations is obtained as the second functional derivative of $S[a,A]$ with respect to a^a_μ, at fixed A^a_μ. This leads to the inverse propagator

$$\mathcal{D}^{ab}_{\mu\nu} = -D^{ac}_\rho D^{cb}_\rho \delta_{\mu\nu} - 2igF^{ab}_{\mu\nu} \tag{3.49}$$

and the effective action

$$\Gamma_{\rm glu}[A] = -\frac{1}{2}\ln\det(\mathcal{D}) = \frac{1}{2}\int_0^\infty \frac{dT}{T}{\rm Tr}\,(e^{-T\mathcal{D}}) \tag{3.50}$$

In writing down eq.(3.49) we have adopted the Feynman gauge $\alpha = 1$. The covariant derivative $D_\mu \equiv \partial_\mu + igA^a_\mu T^a$ and the field strength $F^{ab}_{\mu\nu} \equiv F^c_{\mu\nu}(T^c)^{ab}$ are matrices in the adjoint representation of the gauge group [9]. The full effective action is obtained by adding the contribution

[9] Our definition for the non-abelian covariant derivative is $D_\mu \equiv \partial_\mu + igA^a_\mu T^a$, with $[T^a,T^b] = if^{abc}T^c$. The adjoint representation is given by $(T^a)^{bc} = -if^{abc}$.

of the Faddeev-Popov ghosts to eq. (3.50). The evaluation of the ghost determinant proceeds along the same lines as scalar QED, and will be dealt with later on.

In order to derive a path integral representation of the heat-kernel

$$\mathrm{Tr}\,(\mathrm{e}^{-T\mathcal{D}}) \tag{3.51}$$

we first look at a slightly more general problem. We generalize the operator $\mathcal{D}$ to

$$\widehat{h}_{\mu\nu} \equiv -D^2\delta_{\mu\nu} + M_{\mu\nu}(x) \tag{3.52}$$

where $M_{\mu\nu}(x)$ is an arbitrary matrix in color space. In particular, we do not assume that the Lorentz trace M^μ_μ is zero. Given $M_{\mu\nu}$, we construct the following one-particle Hamilton operator:

$$\widehat{H} = (\widehat{p}_\mu + gA_\mu(\widehat{x}))^2 - :\widehat{\bar{\psi}^\mu} M_{\nu\mu}(\widehat{x})\widehat{\psi}^\nu: \tag{3.53}$$

The system under consideration has a graded phase-space coordinatized by x_μ, p_μ and two sets of anti-commuting variables, ψ_μ and $\bar{\psi}_\mu$, which obey canonical anti-commutation relations:

$$\widehat{\psi}_\mu\widehat{\bar{\psi}}_\nu + \widehat{\bar{\psi}}_\nu\widehat{\psi}_\mu = \delta_{\mu\nu} \tag{3.54}$$

For a reason which will become obvious in a moment we have adopted the "anti-Wick" ordering in (3.53): all $\bar{\psi}$'s are on the right of all ψ's, e.g.

$$\begin{aligned} :\widehat{\psi}_\alpha\widehat{\bar{\psi}}_\beta: &= \widehat{\psi}_\alpha\widehat{\bar{\psi}}_\beta \\ :\widehat{\bar{\psi}}_\beta\widehat{\psi}_\alpha: &= -\widehat{\psi}_\alpha\widehat{\bar{\psi}}_\beta \end{aligned} \tag{3.55}$$

We can represent the commutation relations on a space of wave functions $\Phi(x,\psi)$ depending on x_μ and a set of classical Grassmann variables ψ_μ. The "position" operators $\widehat{x}_\mu = x_\mu$, $\widehat{\psi}_\mu = \psi_\mu$ act multiplicatively on Φ, the conjugate momenta as derivatives $\widehat{p}_\mu = -i\partial_\mu$ and $\widehat{\bar{\psi}}_\mu = \partial/\partial\psi^\mu$. Thus the Hamiltonian becomes [133]

$$\widehat{H} = -D^2 + \psi^\nu M_{\nu\mu}(x)\frac{\partial}{\partial\psi_\mu} \tag{3.56}$$

The wave functions Φ have a decomposition of the form

$$\Phi(x,\psi) = \sum_{p=0}^{D} \frac{1}{p!}\,\phi^{(p)}_{\mu_1\ldots\mu_p}(x)\;\psi^{\mu_1}\ldots\psi^{\mu_p} \tag{3.57}$$

This suggests the interpretation of Φ as an inhomogeneous differential form on $\mathbf{R}^D$ with the fermions ψ^μ playing the role of the differentials dx^μ [134,135]. The form-degree or, equivalently, the fermion number is measured by the operator

$$\widehat{F} = \widehat{\psi}^\mu\widehat{\bar{\psi}}_\mu = \psi^\mu\frac{\partial}{\partial\psi^\mu} \tag{3.58}$$

We are particularly interested in one-forms:

$$\Phi(x,\psi) = \varphi_\mu(x)\psi^\mu \tag{3.59}$$

The Hamiltonian (3.56) acts on them according to

$$(\widehat{H}\Phi)(x,\psi) = (\widehat{h}_\mu{}^\nu\varphi_\nu)\psi^\mu \tag{3.60}$$

We see that, when restricted to the one-form sector, the quantum system with the Hamiltonian (3.53) is equivalent to the one defined by the bosonic matrix Hamiltonian $\widehat{h}_{\mu\nu}$ [133,135]. The Euclidean proper time evolution of the wave functions Φ is implemented by the kernel

$$K(x_2,\psi_2,\tau_2|x_1,\psi_1,\tau_1) = \langle x_2,\psi_2|e^{-(\tau_2-\tau_1)\widehat{H}}|x_1,\psi_1\rangle \tag{3.61}$$

which obeys the Schrödinger equation

$$\left(\frac{\partial}{\partial T}+\widehat{H}\right)K(x,\psi,T|x_0,\psi_0,0) = 0 \tag{3.62}$$

with the initial condition $K(x,\psi,0|x_0,\psi_0,0) = \delta(x-x_0)\delta(\psi-\psi_0)$. It is easy to write down a path integral solution to eq. (3.62). For the trace of K one obtains

$$\mathrm{Tr}\,(e^{-T\widehat{H}_W}) = \int_{\mathcal{P}}\mathcal{D}x(\tau)\int_A \mathcal{D}\psi(\tau)\mathcal{D}\bar{\psi}(\tau)\mathrm{tr}\mathcal{P}e^{-\int_0^T d\tau L} \tag{3.63}$$

with

$$L = \frac{1}{4}\dot{x}^2 + ig\dot{x}^\mu A_\mu + \bar{\psi}^\mu\left(\delta_{\mu\nu}\frac{d}{d\tau} - M_{\nu\mu}\right)\psi^\nu \tag{3.64}$$

(the subscript "W" for H will be explained in a moment). We have again periodic boundary conditions for $x^\mu(\tau)$, and anti-periodic ones for $\psi^\mu(\tau)$.

At this point we have to be careful. If we regard the Hamiltonian (3.53) as a function of the anti-commuting c-numbers ψ_μ and $\bar{\psi}_\mu$ it is related to the classical Lagrangian (3.64) by a standard Legendre transformation. As is well-known, the information about the operator ordering is implicit in the discretization which is used for the definition of the path-integral. Different operator orderings correspond to different discretization prescriptions (see, e.g., [136]). In our derivation of the worldline path integral representation for the spinor loop effective action in the previous section we used the so-called midpoint prescription [137] for the discretization. The reason for this choice is that, by Sato's theorem [138], only in this case the familiar path-integral manipulations are allowed. Those will be needed to justify the naive one-dimensional perturbation expansion which we have in mind.

It is known [137,138,139,140,141] that, at the operator level, this is equivalent to using the Weyl ordered Hamiltonian $\widehat{H}_W$. This is the reason why we wrote $\widehat{H}_W$ rather than $\widehat{H}$ on the l.h.s. of eq. (3.63). In order to arrive at the relation (3.60) we had to assume that the fermion operators in $\widehat{H}$ are "anti-Wick" ordered. Weyl ordering amounts to a symmetrization in $\bar{\psi}$ and ψ so that

$$\begin{aligned}\widehat{H}_W &= (\widehat{p}_\mu + gA_\mu(\widehat{x}))^2 + \frac{1}{2}M_{\nu\mu}(\widehat{x})(\widehat{\psi}^\nu\widehat{\bar{\psi}}^\mu - \widehat{\bar{\psi}}^\mu\widehat{\psi}^\nu)\\ &= \widehat{H} - \frac{1}{2}M^\mu_\mu(\widehat{x})\end{aligned} \tag{3.65}$$

In the second line of (3.65) we used (3.53) and (3.54). (With respect to $\widehat{x}_\mu$ and $\widehat{p}_\mu$ Weyl ordering is used throughout.) If we employ (3.65) in (3.63) we obtain the following representation for the partition function of the anti-Wick ordered exponential:

$$\mathrm{Tr}\,(e^{-T\widehat{H}}) \quad = \quad \int_P \mathcal{D}x(\tau) \int_A \mathcal{D}\psi(\tau)\mathcal{D}\bar{\psi}(\tau)\mathrm{tr}\mathcal{P}\exp\left[-\int_0^T d\tau \left\{L(\tau) + \frac{1}{2}M^\mu_\mu(x(\tau))\right\}\right] \tag{3.66}$$

Let us now calculate the partition function $\mathrm{Tr}\,(\exp(-T\widehat{h}))$ which is a generalization of the heat-kernel needed in eq. (3.50). By virtue of eq. (3.60) we may write

$$\mathrm{Tr}\,(e^{-T\widehat{h}}) = \mathrm{Tr}\,_1(e^{-T\widehat{H}}) \tag{3.67}$$

where "$\mathrm{Tr}\,_1$" denotes the trace in the one-form sector of the theory which contains the worldline fermions. In order to perform the projection on the one-form sector we identify $M_{\mu\nu}$ with

$$M_{\mu\nu} = C\delta_{\mu\nu} - 2igF_{\mu\nu} \tag{3.68}$$

where C is a real constant. As a consequence,

$$\widehat{H} = \widehat{H}_0 + C\widehat{F} \tag{3.69}$$

with

$$\widehat{H}_0 \equiv (\widehat{p}_\mu + gA_\mu(\widehat{x}))^2 - 2igF_{\nu\mu}(\widehat{x})\widehat{\psi}^\nu\widehat{\bar{\psi}^\mu} \tag{3.70}$$

denoting the Hamiltonian which corresponds to the inverse propagator $\mathcal{D}$. The fermion number operator $\widehat{F} \equiv \widehat{\psi}^\mu\widehat{\bar{\psi}}_\mu$ is anti-Wick ordered by definition. Its spectrum consists of the integers $p = 0, 1, 2, \ldots, D$. Note that $M^\mu_\mu = DC$, and that because of the antisymmetry of $F_{\mu\nu}$ the Hamiltonian $\widehat{H}_0$ has no ordering ambiguity in its fermionic piece. It will prove useful to apply eq. (3.66) not to $\widehat{H}$ directly, but to $\widehat{H} - C = \widehat{H}_0 + C(\widehat{F} - 1)$. This leads to

$$\mathrm{Tr}\,(e^{-CT(\widehat{F}-1)}e^{-T\widehat{H}_0}) \quad = \quad \exp\Big[-CT(\frac{D}{2} - 1)\Big] \int_P \mathcal{D}x(\tau) \int_A \mathcal{D}\psi(\tau)\mathcal{D}\bar{\psi}(\tau)\mathrm{tr}\mathcal{P}e^{-\int_0^T d\tau L_{\mathrm{glu}}} \tag{3.71}$$

with

$$L_{\mathrm{glu}} = \frac{1}{4}\dot{x}^2 + ig\dot{x}^\mu A_\mu + \bar{\psi}^\mu\Big[\delta_{\mu\nu}(\frac{d}{d\tau} - C) - 2igF_{\mu\nu}\Big]\psi^\nu \tag{3.72}$$

After having performed the path integration in (3.71) we shall send C to infinity. While this has no effect in the one-form sector, it leads to an exponential suppression factor $\exp[-CT(p-1)]$ in the sectors with fermion numbers $p = 2, 3, ..., D$. Hence only the zero and the one forms survive the limit $C \to \infty$. In order to eliminate the contribution from the zero forms we insert the projector $[1-(-1)^{\widehat{F}}]/2$ into the trace. It projects on the subspace of odd form degrees, and is easily implemented by combining periodic and anti-periodic boundary conditions for ψ_μ. In this way we arrive at the following representation of the partition function of $\widehat{H}_0$, restricted to the one-form sector:

$$
\begin{aligned}
\mathrm{Tr}_1[e^{-T\widehat{H}_0}] &= \lim_{C\to\infty} \mathrm{Tr}\left[\frac{1}{2}(1-(-1)^{\widehat{F}})e^{-CT(\widehat{F}-1)}e^{-T\widehat{H}_0}\right] \\
&= \lim_{C\to\infty} \exp\left[-CT\left(\frac{D}{2}-1\right)\right]\int_P \mathcal{D}x(\tau)\frac{1}{2}(\int_A - \int_P)\mathcal{D}\psi(\tau)\mathcal{D}\bar{\psi}(\tau)\mathrm{tr}\,\mathcal{P}e^{-\int_0^T d\tau L_{\mathrm{glu}}}
\end{aligned}
\tag{3.73}
$$

Because $\mathrm{Tr}\,(\exp(-T\mathcal{D})) = \mathrm{Tr}_1(\exp(-T\widehat{H}_0))$, eq. (3.73) implies for the effective action [64,92]

$$
\begin{aligned}
\Gamma_{\mathrm{glu}}[A] &= \frac{1}{2}\lim_{C\to\infty}\int_0^\infty \frac{dT}{T}\exp\Big[-CT(\frac{D}{2}-1)\Big]\int_P \mathcal{D}x\ \frac{1}{2}(\int_A - \int_P)\mathcal{D}\psi\mathcal{D}\bar{\psi} \\
&\times \mathrm{tr}\,\mathcal{P}\exp\Big[-\int_0^T d\tau\left\{\frac{1}{4}\dot{x}^2 + ig\dot{x}^\mu A_\mu + \bar{\psi}^\mu\Big[\delta_{\mu\nu}(\frac{d}{d\tau}-C) - 2igF_{\mu\nu}\Big]\psi^\nu\right\}\Big]
\end{aligned}
\tag{3.74}
$$

Note that, from the point of view of the worldline fermions, C plays the role of a mass. The factor $\exp[-CTD/2]$ in (3.74) is due to the difference between the Weyl and the anti-Wick-ordered Hamiltonian. It is crucial for obtaining a finite result in the limit $C \to \infty$. In fact, for $D = 4$ it converts the prefactor e^{CT} to a decaying exponential e^{-CT} [10].

A similar worldline path integral representation can also be written down for the gluon propagator in a background Yang-Mills field [92]. This may be useful for future extensions of the string – inspired formalism.

[10] This reordering factor was missing in [64], where the change of the sign in $D = 4$ was instead erroneously attributed to a difference between Minkowski and Euclidean spacetime.

4. Calculation of One-Loop Amplitudes

We proceed to the evaluation of worldline path integrals at the one-loop level. As was already mentioned the method used is a very specific one, and analogous to the techniques used in string perturbation theory. All path integrals will be manipulated into Gaussian form, which reduces their evaluation to the calculation of worldline propagators and determinants, and standard combinatorics. Of course there exist many alternatives to this procedure (see, e.g., [48,38,50,52,44,47,53,45]). Those will not be discussed here.

While most of the formalism developed here applies to an arbitrary spacetime dimension, or at least to even dimensions, in this review all of our applications will be to four dimensional field theories (for some calculations in $D = 2$ see [103], in $D = 3$ [142]).

4.1. The N - point Amplitude in Scalar Field Theory

At the one-loop level, the worldline formalism has been used for a large variety of purposes. Let us begin with the simplest possible case, the one-loop N - point amplitude in massive ϕ^3 - theory. Choosing

$$U(\phi) = \frac{\lambda}{3!}\phi^3 \tag{4.1}$$

in eq.(3.9), the path integral for the corresponding effective action reads

$$\Gamma[\phi] = \frac{1}{2}\int_0^\infty \frac{dT}{T}\, e^{-m^2 T} \int_{x(T)=x(0)} \mathcal{D}x(\tau)\, e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2 + \lambda\phi(x(\tau))\right)} \tag{4.2}$$

We intend to calculate this path integral using the elementary Gauss formula

$$\int dx\, \mathrm{e}^{-x\cdot A\cdot x + 2b\cdot x} \sim (\det(A))^{-\frac{1}{2}} \mathrm{e}^{\,b\cdot A^{-1}\cdot b} \tag{4.3}$$

As we already explained in the introduction, first one has to deal with the zero-mode contained in the coordinate path integral $\int \mathcal{D}x(\tau)$, i.e. the constant loops. This is done by separating off the integration over the loop center of mass x_0, which reduces the coordinate path integral to an integral over the relative coordinate y:

$$\begin{aligned} \int \mathcal{D}x &= \int dx_0 \int \mathcal{D}y \\ x^\mu(\tau) &= x_0^\mu + y^\mu(\tau) \\ \int_0^T d\tau\, y^\mu(\tau) &= 0 \end{aligned} \tag{4.4}$$

The effective action thereby gets expressed in terms of an effective Lagrangian $\mathcal{L}_{eff}$,

$$\Gamma = \int dx_0\, \mathcal{L}_{eff}(x_0) \tag{4.5}$$

and $\mathcal{L}_{eff}(x_0)$ is represented as an integral over the space of all loops with fixed common center of mass x_0.

In the reduced Hilbert space without the zero mode, the kinetic operator is invertible, and the inverse is easily found using the eigenfunctions of the derivative operator on the circle with circumference T, $\{e^{2\pi i n \frac{\tau}{T}}, n \in \mathbf{Z}\backslash\{0\}\}$:

$$2\langle\tau_1 \mid (\frac{d}{d\tau})^{-2} \mid \tau_2\rangle = 2T \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{2\pi i n \frac{\tau_1-\tau_2}{T}}}{(2\pi i n)^2} = \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T} - \frac{T}{6} \tag{4.6}$$

($\tau_1 - \tau_2 \in [-T, T]$). It will be seen later on that the constant $-T/6$ drops out of all physical results, so that we can delete it at the beginning. The remainder is the "bosonic" Green's function which we already introduced in eq.(1.16),

$$G_B(\tau_1,\tau_2) = \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T}$$

Note that it is continuous as a function on $S^1 \times S^1$. Its value depends neither on the location of the zero on the circle, nor on the choice of orientation. This Green's function we will always use as the correlator for the coordinate "field",

$$\langle y^\mu(\tau_1) y^\nu(\tau_2)\rangle = -g^{\mu\nu} G_B(\tau_1,\tau_2) \tag{4.7}$$

The zero mode fixing prescription, and consequently also the form of this worldline correlator, are not unique [106,107,143,144]. This ambiguity is of some technical importance, and will be discussed in chapter 7 in connection with the calculation of the effective action itself.

The only other information required for the execution of a Gaussian path integral is the free path integral determinant. With our conventions, the free coordinate path integral at fixed proper-time T is

$$\int \mathcal{D}y \exp\Bigl[-\int_0^T d\tau \frac{1}{4}\dot{y}^2\Bigr] = (4\pi T)^{-\frac{D}{2}} \tag{4.8}$$

Here the T - dependence can be easily determined by, e.g., ζ - function regularization [121], while the factor $[4\pi]^{-\frac{D}{2}}$ corresponds to the usual loop-counting factor in quantum field theory.

How to continue now depends on whether we wish to compute the effective action itself, or the corresponding scattering amplitudes. For the calculation of the effective action, one way to proceed after the expansion of the interaction exponential is to Taylor-expand the external field at the loop center of mass x_0,

$$\phi(x) = e^{y\cdot\partial}\phi(x_0) \tag{4.9}$$

As we will see in chapter 7, this leads to a highly efficient algorithm [96,106,107,143] for calculating derivative expansions of effective actions.

At the moment we are interested in the calculation of the N - point amplitude, which proceeds somewhat differently. According to standard quantum field theory (see, e.g., [117]), the (one-particle-irreducible) N – point function can be obtained from the one-loop effective action $\Gamma[\phi]$

by a N - fold functional differentiation with respect to ϕ. In x – space, we can implement this operation simply by expanding the interaction exponential to N - th order, and inserting appropriate δ – functions into the path integral [63]:

$$
\begin{aligned}
\Gamma_{\rm 1PI}[x_1,\dots,x_N] &= \frac{1}{2}(-\lambda)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int \mathcal{D}x \\
&\quad \times \int_0^T \prod_{i=1}^N d\tau_i\, \delta\Bigl(x(\tau_i)-x_i\Bigr) e^{-\int_0^T d\tau \left(\frac{1}{4}\dot x^2\right)}
\end{aligned}
\tag{4.10}
$$

Thus only trajectories running through the prescribed points $x_1,\dots,x_N$ will contribute to the amplitude. The subscript "1PI" stands for one-particle-irreducible, and needs to be introduced here since, in contrast to the N - photon amplitude treated in the introduction, the N - point function in ϕ^3 - theory has also one-particle-reducible contributions.

Similarly, the N – point function in momentum space can be obtained by specializing the background to a sum of plane waves,

$$
\phi(x) = \sum_{i=1}^N \mathrm{e}^{ip_i\cdot x} \tag{4.11}
$$

Then one picks out the term containing every p_i precisely once (compare eqs.(1.11), (1.12)). This leads to

$$
\begin{aligned}
\Gamma_{\rm 1PI}[p_1,\dots,p_N] &= \frac{1}{2}(-\lambda)^N \int_0^\infty \frac{dT}{T}\, e^{-m^2T} \int_0^T \prod_{i=1}^N d\tau_i \int dx_0 \int \mathcal{D}y \\
&\quad \times \exp\Bigl[i\sum_{i=1}^N p_i\cdot x(\tau_i)\Bigr] e^{-\int_0^T d\tau \left(\frac{1}{4}\dot x^2\right)}
\end{aligned}
\tag{4.12}
$$

Note that now every external leg is represented by a scalar vertex operator, eq.(1.3) [11]. Since $x_i \equiv x(\tau_i) = x_0 + y(\tau_i)$, the x_0 - integral just gives momentum conservation,

$$
\int dx_0 \exp\Bigl[ix_0\cdot \sum_{i=1}^N p_i\Bigr] = (2\pi)^D \delta\Bigl(\sum_{i=1}^N p_i\Bigr) \tag{4.13}
$$

The y - path integral is now Gaussian, and can be simply calculated by "completing the square". One obtains the following parameter integral,

$$
\begin{aligned}
\Gamma_{\rm 1PI}[p_1,\dots,p_N] &= \frac{1}{2}(-\lambda)^N (2\pi)^D \delta\bigl(\sum p_i\bigr) \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2T} \\
&\quad \times \prod_{i=1}^N \int_0^T d\tau_i \exp\Bigl[\frac{1}{2}\sum_{i,j=1}^N G_B(\tau_i,\tau_j)p_i\cdot p_j\Bigr]
\end{aligned}
\tag{4.14}
$$

[11] In our present conventions momenta appearing in vertex operators are *ingoing*.

This representation of the one-loop N-point amplitude in ϕ^3 – theory appears, as far as is known to the author, first in [63]. It is the simplest example of a Bern-Kosower type formula. Note that a constant added to G_B would drop out immediately on account of momentum conservation.

Using momentum conservation this parameter integral can, for any given ordering of the external legs along the loop, be readily transformed into the corresponding standard Feynman parameter integral [21,24,145,146].

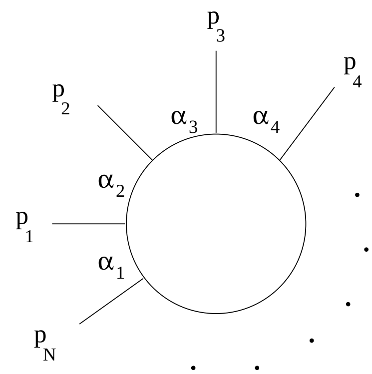


Figure 12: One-loop N-point diagram.

To obtain the contribution of the Feynman diagram with the standard ordering of the momenta $p_1, \ldots, p_N$ (fig. 12), one simply restricts the τ - integrations to the sector defined by $T \geq \tau_1 \geq \tau_2 \geq \ldots \tau_N = 0$, and transforms from τ - parameters to α - parameters:

$$
\begin{aligned}
\alpha_1 &= T - \tau_1 \\
\alpha_2 &= \tau_1 - \tau_2 \\
\cdots & \quad \cdots \\
\alpha_N &= \tau_{N-1}
\end{aligned}
\tag{4.15}
$$

Here we have made use of the freedom to choose the zero somewhere on the "worldloop" for setting $\tau_N = 0$. This is always possible, since our worldline Green's function $G_B(\tau_1, \tau_2)$ is translation invariant in τ. The complete parameter integral represents the sum of that particular Feynman diagram together with all the "crossed" ones.

The case of ϕ^4 – theory is only marginally different at the one-loop level. A field theory potential of $U(\phi) = \frac{\lambda}{4!}\phi^4$ leads to a worldline interaction Lagrangian of $L_{\text{int}} = \frac{\lambda}{2}\phi^2$. The formula for the $2N$ – point amplitude analogous to eq.(4.14) is

$$
\begin{aligned}
\Gamma_{\text{1PI}}[p_1, \ldots, p_{2N}] &= \frac{1}{2}(-\lambda)^N \int_0^\infty \frac{dT}{T} e^{-m^2 T} \int_0^T \prod_{i=1}^N d\tau_i \int dx_0 \int \mathcal{D}y \\
&\quad \times \exp\Bigl[i \sum_{i=1}^N (p_{2i-1} + p_{2i}) \cdot x(\tau_i) \Bigr] e^{-\int_0^T d\tau \left(\frac{1}{4}\dot{x}^2\right)} + \text{permuted terms}
\end{aligned}
\tag{4.16}
$$

Here one must explicitly sum over all possible ways of partitioning the $2N$ external states into N pairs.

4.2. Photon Scattering in Quantum Electrodynamics

Since the scalar loop contribution to the one-loop N – photon scattering amplitude has already been discussed in the introduction, we immediately turn to the spinor loop case. The appropriate path integral representation for the one-loop effective action was given in eq.(1.9),

$$\begin{aligned}\Gamma_{\rm spin}[A] &= -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2 T}\int_P \mathcal{D}x \int_A \mathcal{D}\psi \\ &\quad\times\exp\Bigl[-\int_0^T d\tau\Bigl(\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi}+ieA\cdot\dot{x}-ie\psi\cdot F\cdot\psi\Bigr)\Bigr]\end{aligned} \tag{4.17}$$

Besides the periodic coordinate functions $x^\mu(\tau)$ we now need to also integrate over the $\psi^\mu(\tau)$'s, which are anti-periodic Grassmann functions. As was explained before, the scalar loop case is obtained from this simply by discarding all Grassmann quantities. As a consequence, in this formalism all calculations performed in fermion QED include the corresponding scalar QED results as a byproduct (as far as the calculation of the bare regularized amplitudes is concerned).

Thus the calculation of the x - path integral proceeds as before. For the ψ - path integral, first note that there is no zero-mode due to the anti-periodicity. To find the appropriate worldline Green's function, we now need to invert the first derivative in the Hilbert space of anti-periodic functions. This yields

$$2\langle\tau_1 \mid (\frac{d}{d\tau})^{-1} \mid \tau_2\rangle = 2\sum_{n=-\infty}^{\infty}\frac{e^{2\pi i(n+\frac{1}{2})\frac{\tau_1-\tau_2}{T}}}{2\pi i(n+\frac{1}{2})} = \mathrm{sign}(\tau_1-\tau_2) \equiv G_F(\tau_1,\tau_2) \tag{4.18}$$

($\tau_1 - \tau_2 \in [T, -T]$). Thus we have now the following two Wick contraction rules,

$$\begin{aligned}\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle &= -g^{\mu\nu}G_B(\tau_1,\tau_2) = &-g^{\mu\nu}\Bigl[\mid\tau_1-\tau_2\mid-\frac{(\tau_1-\tau_2)^2}{T}\Bigr]\\ \langle\psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}g^{\mu\nu}G_F(\tau_1,\tau_2) = &\frac{1}{2}g^{\mu\nu}\mathrm{sign}(\tau_1-\tau_2)\end{aligned} \tag{4.19}$$

With our present conventions the free ψ - path integral is normalized as [12]

$$\int\mathcal{D}\psi\exp\Bigl[-\int_0^T d\tau\frac{1}{2}\psi\cdot\dot{\psi}\Bigr] = 4 \tag{4.20}$$

This takes into account the fact that a Dirac spinor in four dimensions has four real degrees of freedom.

[12] This convention differs from the one used in [96,84,92].

One–loop scattering amplitudes are again obtained by specializing the background to a finite sum of plane waves of definite polarization. Equivalently one introduces a photon vertex operator V^A representing an external photon of definite momentum and polarization (compare eq.(1.4)). For the spinor loop case this vertex operator is

$$V^A_{\rm spin}[k,\varepsilon] = \int_0^T d\tau \Big[\varepsilon\cdot\dot{x} + 2i\varepsilon\cdot\psi k\cdot\psi\Big]\,{\rm e}^{ik\cdot x} \tag{4.21}$$

The photon vertex operator $V^A_{\rm scal}[k,\varepsilon]$ for the scalar loop is given by the same expression without the Grassmann term. We can then express the scalar and spinor QED N - photon amplitudes in terms of Wick contractions of vertex operators as follows,

$$\Gamma_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] = (-ie)^N\int_0^\infty \frac{dT}{T}{\rm e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Big\langle V^A_{{\rm scal},1}\cdots V^A_{{\rm scal},N}\Big\rangle \tag{4.22}$$

$$\Gamma_{\rm spin}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] = -2(-ie)^N\int_0^\infty \frac{dT}{T}{\rm e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Big\langle V^A_{{\rm spin},1}\cdots V^A_{{\rm spin},N}\Big\rangle \tag{4.23}$$

Here the zero-mode integration has already been performed, and the resulting factor (4.13) been omitted. The normalization refers to the complex scalar and Dirac fermion cases.

The bosonic Wick contractions may be performed using the formal exponentiation as explained in the introduction, eq.(1.17), leading to the Bern-Kosower master formula eq.(1.18). Alternatively, one may follow the following simple general rules for the Wick contraction of expressions involving both elementary fields and exponentials:

1. Contract fields with each other as usual, and fields with exponentials according to

$$\Big\langle y^\mu(\tau_1)\,{\rm e}^{ik\cdot y(\tau_2)}\Big\rangle = i\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle k_\nu {\rm e}^{ik\cdot y(\tau_2)} \tag{4.24}$$

   (the field disappears, the exponential stays in the game).

2. Once all elementary fields have been eliminated, the contraction of the remaining exponentials yields a universal factor

$$\Big\langle {\rm e}^{ik_1\cdot y_1}\cdots {\rm e}^{ik_N\cdot y_N}\Big\rangle = \exp\Big[-\frac{1}{2}\sum_{i,j=1}^N k_{i\mu}\langle y^\mu(\tau_i)y^\nu(\tau_j)\rangle k_{j\nu}\Big] \tag{4.25}$$

Since the photon vertex operator eq.(4.21) contains $\dot{x}$ we will also need to Wick-contract expressions involving $\dot{y}$. It is always assumed that Wick-contractions commute with derivatives. Therefore the first and second derivatives of G_B will appear,

$$\begin{aligned}\dot{G}_B(\tau_1,\tau_2) &= {\rm sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T}\\ \ddot{G}_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}\end{aligned} \tag{4.26}$$

Turning our attention to the ψ – path integral, first note that its explicit execution would be algebraically equivalent to the calculation of the corresponding Dirac traces in field theory. For example, the correlator of four ψ's gives

$$\Bigl\langle \psi^\kappa(\tau_1)\psi^\lambda(\tau_2)\psi^\mu(\tau_3)\psi^\nu(\tau_4)\Bigr\rangle = \frac{1}{4}\Bigl[G_{F12}G_{F34}g^{\kappa\lambda}g^{\mu\nu} - G_{F13}G_{F24}g^{\kappa\mu}g^{\lambda\nu} + G_{F14}G_{F23}g^{\kappa\nu}g^{\lambda\mu}\Bigr] \tag{4.27}$$

Choosing a definite ordering of the proper-time arguments, e.g. $\tau_1 > \tau_2 > \tau_3 > \tau_4$, we have the familiar alternating sum of products of metric tensors at hand which appears also in the trace of the product of four Dirac matrices.

However, the explicit computation of the ψ - integral can be avoided, due to the following remarkable feature of the Bern-Kosower formalism. After the evaluation of the x – path integral, one is left with an integral over the parameters $T, \tau_1, \ldots, \tau_N$, where N is the number of external legs. The integrand is an expression consisting of the ubiquitous exponential factor $\exp\Bigl[\frac{1}{2}\sum_{i,j=1}^N G_B(\tau_i,\tau_j)k_i \cdot k_j\Bigr]$ multiplied by a prefactor P_N which is a polynomial function of the $\dot{G}_{Bij}$'s and $\ddot{G}_{Bij}$'s, as well as of the kinematic invariants.

As Bern and Kosower have shown in appendix B of [20], all the $\ddot{G}_{Bij}$'s can be eliminated by suitable chains of partial integrations in the parameters τ_i, leading to an equivalent parameter integral involving only the G_{Bij}'s and $\dot{G}_{Bij}$'s. According to the Bern-Kosower rules for the spinor loop case, all contributions from fermionic Wick contractions may then be taken into account simply by simultaneously replacing every closed cycle of $\dot{G}_B$'s appearing, say $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}$, by its "supersymmetrization",

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \rightarrow \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1} \tag{4.28}$$

(see eq.(2.15)).

The validity of this rule can be understood either in terms of worldsheet [66] or worldline [64] supersymmetry. We will give a direct combinatorial proof in appendix D, using the worldline superfield formalism introduced in section 3.3 and results from section 4.8 below. The rule reduces the transition from the scalar to the spinor loop case to a mere pattern matching problem.

Yet another option in the spinor loop calculation is the explicit use of the superfield formalism. If one takes the super path integral representation of the effective action, eq. (3.41), as a starting point in the construction of the photon scattering amplitude, one still obtains eq.(4.23). However the photon vertex operator then appears rewritten as

$$V^A_{\rm spin}[k,\varepsilon] = \int_0^T d\tau \int d\theta\, \varepsilon \cdot DX \exp[ik \cdot X] \tag{4.29}$$

This allows one to combine the two Wick contraction rules eqs.(4.19) into a single one,

$$\langle Y^\mu(\tau_1,\theta_1)Y^\nu(\tau_2,\theta_2)\rangle = -g^{\mu\nu}\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2) \tag{4.30}$$

with a worldline superpropagator

$$\hat{G}(\tau_1,\theta_1;\tau_2,\theta_2) \equiv G_B(\tau_1,\tau_2) + \theta_1\theta_2 G_F(\tau_1,\tau_2) \tag{4.31}$$

One can then write down a master formula for N – photon scattering [65] which is formally analogous to the one for the scalar loop, eq.(1.18),

$$\begin{aligned}\Gamma_{\rm spin}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ie)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\\ &\times\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i \exp\Bigl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j+iD_i\hat{G}_{ij}\varepsilon_i\cdot k_j+\frac{1}{2}D_iD_j\hat{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Bigr]\Bigr\}\,|_{\varepsilon_1\dots\varepsilon_N}\end{aligned} \tag{4.32}$$

Here, as well as in (4.29), we introduce the further convention that also the polarization vectors $\varepsilon_1,\dots,\varepsilon_N$ are to be treated as Grassmann variables. Thus we have now all ψ's, θ's, $d\theta$'s, and ε's anticommuting with each other. The overall sign of the master formula refers to the standard ordering of the polarization vectors $\varepsilon_1\varepsilon_2\dots\varepsilon_N$ [13].

The superfield formalism thus also avoids the explicit execution of the Grassmann - Wick contractions. Those are now replaced by a number of Grassmann integrations, which have to be performed at a later stage of the calculation. Ultimately the superfield formalism leads to the same collection of parameter integrals to be performed as the component formalism, however it is often useful for keeping intermediate expressions compact.

4.3. Example: QED Vacuum Polarization

As a first example, let us recalculate in detail the one-loop vacuum polarization tensors in scalar and spinor QED [64].

4.3.1. Scalar QED

According to the above the one-loop two-photon amplitude in scalar QED can be written as

$$\begin{aligned}\Gamma^{\mu\nu}_{\rm scal}[k_1,k_2] &= (-ie)^2\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}\int\mathcal{D}x\int_0^T d\tau_1\int_0^T d\tau_2\\ &\times\dot{x}^\mu(\tau_1)\mathrm{e}^{ik_1\cdot x(\tau_1)}\dot{x}^\nu(\tau_2)\mathrm{e}^{ik_2\cdot x(\tau_2)}\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot{x}^2}\end{aligned} \tag{4.33}$$

Separating off the zero mode according to eqs.(4.4),(4.13), one obtains

$$\begin{aligned}\Gamma^{\mu\nu}_{\rm scal}[k_1,k_2] &= (2\pi)^D\delta(k_1+k_2)\,\Pi^{\mu\nu}_{\rm scal}(k_1)\\ \Pi^{\mu\nu}_{\rm scal}(k) &= -e^2\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}\int_0^T d\tau_1\int_0^T d\tau_2\\ &\times\int\mathcal{D}y\,\dot{y}^\mu(\tau_1)\mathrm{e}^{ik\cdot y(\tau_1)}\dot{y}^\nu(\tau_2)\mathrm{e}^{-ik\cdot y(\tau_2)}\mathrm{e}^{-\int_0^T d\tau\frac{1}{4}\dot{y}^2}\end{aligned} \tag{4.34}$$

[13] With our conventions a Wick rotation $k_i^4\to -ik_i^0, T\to is$ yields the N - photon amplitude in the conventions of [136].

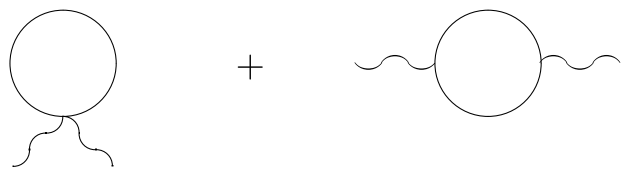


Figure 13: Scalar QED vacuum polarization diagrams.

The Wick contraction of the two photon vertex operators according to the above rules produces two terms,

$$\Big\langle \dot{y}^\mu(\tau_1)\mathrm{e}^{ik\cdot y(\tau_1)}\dot{y}^\nu(\tau_2)\mathrm{e}^{-ik\cdot y(\tau_2)}\Big\rangle = \Big\{ g^{\mu\nu}\ddot{G}_{B12} - k^\mu k^\nu \dot{G}^2_{B12}\Big\}\,\mathrm{e}^{-k^2 G_{B12}} \tag{4.35}$$

Now one could just write out G_B and its derivatives, and perform the parameter integrals. It turns out to be useful, though, to first remove the $\ddot{G}_{B12}$ appearing in the first term by a partial integration in the variable τ_1 or τ_2. The integrand then turns into

$$\Big\{ g^{\mu\nu}k^2 - k^\mu k^\nu\Big\}\dot{G}^2_{B12}\mathrm{e}^{-k^2 G_{B12}} \tag{4.36}$$

Note that this makes the transversality of the vacuum polarization tensor manifest. We rescale to the unit circle, $\tau_i = Tu_i, i = 1,2$, and use the translation invariance in τ to fix the zero to be at the location of the second vertex operator, $u_2 = 0, u_1 = u$. We have then

$$\begin{aligned} G_B(\tau_1,\tau_2) &= Tu(1-u) \\ \dot{G}_B(\tau_1,\tau_2) &= 1-2u \end{aligned} \tag{4.37}$$

Taking the free determinant factor eq.(4.8) into account, and performing the global proper-time integration, one finds

$$\begin{aligned} \Pi^{\mu\nu}_{\mathrm{scal}}(k) &= e^2\Big[k^\mu k^\nu - g^{\mu\nu}k^2\Big]\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}T^2\int_0^1 du(1-2u)^2\mathrm{e}^{-Tu(1-u)k^2} \\ &= \frac{e^2}{(4\pi)^{\frac{D}{2}}}\Big[k^\mu k^\nu - g^{\mu\nu}k^2\Big]\Gamma(2-\frac{D}{2})\int_0^1 du(1-2u)^2\Big[m^2+u(1-u)k^2\Big]^{\frac{D}{2}-2} \end{aligned} \tag{4.38}$$

The reader is invited to verify that this agrees with the result reached by calculating, in dimensional regularization, the sum of the corresponding two Feynman diagrams (fig. 13).

4.3.2. Spinor QED

For the fermion loop the path integral for the two-photon amplitude becomes, in the component formalism,

$$\begin{aligned}\Gamma^{\mu\nu}_{\rm spin}[k_1,k_2] &= -\frac{1}{2}(-ie)^2\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}\int\mathcal{D}x\int\mathcal{D}\psi\int_0^T d\tau_1\int_0^T d\tau_2\\ &\quad\times\Bigl(\dot x_1^\mu+2i\psi_1^\mu\psi_1\cdot k_1\Bigr)\mathrm{e}^{ik_1\cdot x_1}\Bigl(\dot x_2^\nu+2i\psi_2^\nu\psi_2\cdot k_2\Bigr)\mathrm{e}^{ik_2\cdot x_2}\mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot x^2+\frac{1}{2}\psi\dot\psi\right)}\end{aligned}$$

The calculation of $\mathcal{D}x$ is identical with the scalar QED calculation. Only the calculation of $\mathcal{D}\psi$ is new, and amounts to a single Wick contraction,

$$(2i)^2\Bigl\langle\psi_1^\mu\psi_1\cdot k_1\psi_2^\nu\psi_2\cdot k_2\Bigr\rangle = G_{F12}^2\Bigl[g^{\mu\nu}k_1\cdot k_2-k_2^\mu k_1^\nu\Bigr] \tag{4.39}$$

Adding this term to the bosonic result shows that, up to the global normalization, the parameter integral for the spinor loop is obtained from the one for the scalar loop simply by replacing, in eq.(4.36),

$$\dot G_{B12}^2\to\dot G_{B12}^2-G_{F12}^2 \tag{4.40}$$

This is in accordance with eq.(2.15). The complete change thus amounts to supplying eq.(4.38) with a global factor of -2, and replacing $(1-2u)^2$ by $-4u(1-u)$. This leads to

$$\Pi^{\mu\nu}_{\rm spin}(k) = 8\frac{e^2}{(4\pi)^{\frac{D}{2}}}\Bigl[k^\mu k^\nu-g^{\mu\nu}k^2\Bigr]\Gamma(2-\frac{D}{2})\int_0^1du\,u(1-u)\Bigl[m^2+u(1-u)k^2\Bigr]^{\frac{D}{2}-2} \tag{4.41}$$

again in agreement with the result of the standard textbook calculation.

4.4. Scalar Loop Contribution to Gluon Scattering

As we discussed in section 3.4.1, the path integral representing the effective action for the scalar loop in a gluon field differs from the photon case, eq.(1.8), only by the path-ordering of the exponentials, and the addition of a global color trace. The gluon vertex operator eq.(1.4) therefore differs from the photon vertex operator eq.(4.21) only by the additional T^a factor, which denotes a gauge group generator in the representation of the loop particle. However, the path-ordering (= proper-time ordering = color ordering) of the path integral has the effect that now those N vertex operators appear inserted on the worldloop in a fixed ordering. Thus the global color trace factors out, and the scalar loop Bern-Kosower master formula eq.(1.18) generalizes to the gluon scattering case as follows,

$$\begin{aligned}\Gamma^{a_1\dots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= (-ig)^N\mathrm{tr}(T^{a_1}\cdots T^{a_N})(2\pi)^D\delta(\sum k_i)\\ &\quad\times\int_0^\infty dT(4\pi T)^{-\frac{D}{2}}e^{-m^2T}\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\dots\int_0^{\tau_{N-2}}d\tau_{N-1}\\ &\quad\times\exp\Bigl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}G_{Bij}k_i\cdot k_j-i\dot G_{Bij}\varepsilon_i\cdot k_j+\frac{1}{2}\ddot G_{Bij}\varepsilon_i\cdot\varepsilon_j\Bigr]\Bigr\}\Big|_{\rm multi-linear}\end{aligned} \tag{4.42}$$

Here we have already eliminated one integration by setting $\tau_N = 0$. Note that it can now happen that a $\delta(\tau_i - \tau_{i+1})$, generated by the Wick contractions, appears multiplied by a $\theta(\tau_i - \tau_{i+1})$, generated by the path–ordering. Symmetry then dictates that just one half of this δ – function should be allowed to contribute to the color ordering under consideration.

The interpretation of this non-abelian master integral differs from its abelian counterpart in two important ways. Firstly, note that in the abelian case we can construct the complete amplitude in either of two ways. We can calculate the integral in the ordered sector $\tau_1 > \tau_2 > \ldots > \tau_N = 0$, and then generate the "crossed" contributions by explicit permutations of the result. Alternatively, we can generate those permuted terms by including the other ordered sectors in the integration. The second option does not exist in the non-Abelian case, since the crossed terms now generally have different color traces.

Secondly, in contrast to the one-loop photon scattering amplitudes the gluonic amplitudes generally have one-particle reducible contributions in addition to the irreducible ones. Since our derivation of the above master formula was based on a path integral representing the one-loop effective action for the gluon field, which is the generator for the one-particle irreducible gluon correlators, the master formula as it stands yields precisely the contributions of all one-particle irreducible graphs to the amplitude in question. To complete the construction of the full one - loop N - gluon amplitude inside field theory one would now have to generate the missing diagrams by an explicit Legendre transformation, amounting to sewing trees onto the one-particle irreducible diagrams. While feasible [66], this would to some extent spoil the elegance of the string-inspired approach. Fortunately, we have seen already in chapter two that, as far as the on-shell amplitude is concerned, the full set of string-derived rules tells us how to bypass this procedure. Steps 3 to 5 of the rules instruct us, starting from the master formula, to remove all second derivatives $\ddot{G}_B$'s, and then to apply the tree replacement ("pinch") rules. In this way all the one-particle-reducible terms are included automatically. This is a remnant of the fact that the fragmentation of the amplitude into one-particle reducible and irreducible diagrams appears only in the infinite string tension limit, when parts of the string worldsheet get "pinched off" [21,66].

4.5. Spinor Loop Contribution to Gluon Scattering

As we have already seen in the previous chapter, the fermion loop case is more involved. In the non-abelian case, the worldline Lagrangian eq.(1.9) now contains a term $\psi^\mu[A_\mu, A_\nu]\psi^\nu$. In the component formalism this would force one to introduce, besides the one-gluon vertex operator

$$V^A_{\rm spin}[k,\varepsilon,a] = T^a \int_0^T d\tau \Bigl(\varepsilon\cdot\dot{x} + 2i\varepsilon\cdot\psi k\cdot\psi\Bigr)\exp[ik\cdot x] \tag{4.43}$$

an additional two – gluon vertex operator [64,94]. This is not necessary in the superfield formalism, where the single gluon super vertex operator

$$V^A_{\rm spin}[k,\varepsilon,a] = T^a \int_0^T d\tau d\theta\, \varepsilon\cdot DX \exp[ik\cdot X] \tag{4.44}$$

remains sufficient. This is because, as explained in section 3.4, the above commutator terms are then generated automatically by the super θ – functions implicit in the path-ordering. Our eqs.(4.22),(4.23) for the N-point functions in the abelian case thus generalize to the non-abelian case simply as follows,

$$
\begin{aligned}
\Gamma^{a_1\dots a_N}_{\text{1PI,scal}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= (-ig)^N \text{tr}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Big\langle V^A_{\text{scal},1}\cdots V^A_{\text{scal},N}\Big\rangle \\
&\quad \times\delta(\frac{\tau_N}{T})\prod_{i=1}^{N-1}\theta(\tau_i-\tau_{i+1})
\end{aligned}
\tag{4.45}
$$

$$
\begin{aligned}
\Gamma^{a_1\dots a_N}_{\text{1PI,spin}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -2(-ig)^N \text{tr}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\Big\langle V^A_{\text{spin},1}\cdots V^A_{\text{spin},N}\Big\rangle \\
&\quad \times\delta(\frac{\tau_N}{T})\prod_{i=1}^{N-1}\theta(\hat{\tau}_{i(i+1)})
\end{aligned}
\tag{4.46}
$$

Here it is understood that V_i^A carries a T^{a_i}.

4.6. Gluon Loop Contribution to Gluon Scattering

In section 3.4 we derived the following path integral representation of the one-loop effective action in pure Yang-Mills theory, using the background field method in Feynman gauge,

$$
\begin{aligned}
\Gamma_{\text{glu}}[A] &= \frac{1}{2}\lim_{C\to\infty}\int_0^\infty\frac{dT}{T}\exp\Big[-CT(\frac{D}{2}-1)\Big]\int_P\mathcal{D}x\,\frac{1}{2}\Big(\int_A-\int_P\Big)\mathcal{D}\psi\mathcal{D}\bar{\psi} \\
&\quad\times\text{tr}\,\mathcal{P}\exp\Big\{-\int_0^T d\tau\Big[\frac{1}{4}\dot{x}^2+ig\dot{x}^\mu A_\mu+\bar{\psi}^\mu\Big[(\frac{d}{d\tau}-C)\delta_{\mu\nu}-2igF_{\mu\nu}\Big]\psi^\nu\Big]\Big\}
\end{aligned}
$$

Recall that this path integral describes a whole multiplet of $p-$ forms, $p=0,\dots,D$ circulating in the loop; the role of the limit $C\to\infty$ is to suppress all contributions from $p\geq 2$, and the contributions from the zero form cancel out in the combination $\int_A-\int_P$.

The Grassmann path integral now appears both with anti-periodic and periodic boundary conditions. The worldline Green's function to be used for its evaluation is [92]

$$
G^C(\tau_1,\tau_2)\equiv\langle\tau_1|(\frac{d}{d\tau}-C)^{-1}|\tau_2\rangle \tag{4.47}
$$

and reads for periodic and anti-periodic boundary conditions, respectively,

$$
G^C_P(\tau_1,\tau_2)=-\Big[\theta(\tau_2-\tau_1)+\theta(\tau_1-\tau_2)e^{-CT}\Big]\frac{e^{C(\tau_1-\tau_2)}}{1-e^{-CT}} \tag{4.48}
$$

$$
G^C_A(\tau_1,\tau_2)=-\Big[\theta(\tau_2-\tau_1)-\theta(\tau_1-\tau_2)e^{-CT}\Big]\frac{e^{C(\tau_1-\tau_2)}}{1+e^{-CT}} \tag{4.49}
$$

We observe that for $C\to\infty$ there is an increasingly strong asymmetry between the forward and backward propagation in the proper time. The derivation of these Green's functions is given in appendix B.

The representation (4.47) of the effective action does not coincide with the one used by Strassler [64]. While he uses the same kinetic term in the fermionic worldline Lagrangian, he modifies

the interaction term according to

$$\bar{\psi}^\mu F_{\mu\nu}\psi^\nu \to \frac{1}{2}\chi^\mu F_{\mu\nu}\chi^\nu \equiv \bar{\psi}^\mu F_{\mu\nu}\psi^\nu + \frac{1}{2}F_{\mu\nu}(\psi^\mu\psi^\nu + \bar{\psi}^\mu\bar{\psi}^\nu) \tag{4.50}$$

where

$$\chi^\mu(\tau) \equiv \psi^\mu(\tau) + \bar{\psi}^\mu(\tau) \tag{4.51}$$

After this modification, Wick contractions involve the $2-$ point function of χ, i.e.

$$\langle \chi^\mu(\tau_1)\chi^\nu(\tau_2)\rangle = g^{\mu\nu}G^\chi(\tau_1,\tau_2) \tag{4.52}$$

where

$$G^\chi(\tau_1,\tau_2) \equiv G^C(\tau_1,\tau_2) - G^C(\tau_2,\tau_1) \tag{4.53}$$

From (4.48), (4.49) we obtain explicitly

$$\begin{aligned} G_P^\chi(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)\frac{\sinh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\sinh[CT/2]} \\ G_A^\chi(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2)\frac{\cosh[C(\frac{T}{2}-|\tau_1-\tau_2|)]}{\cosh[CT/2]} \end{aligned} \tag{4.54}$$

These Green's functions still do not quite coincide with the ones given by Strassler [64]; however, they become effectively equivalent in the limit $C \to \infty$. The modified version is more convenient for the calculation of scattering amplitudes.

Although the worldline Lagrangians for the gluon and for the spinor loop are similar in structure, there is no analogue of the supersymmetry transformations eq.(1.10) in the spin $-$ 1 case. As was noted in [94], it is useful nevertheless to introduce the superfield formalism as a book-keeping device. In complete analogy to the spinor loop case we introduce new superfields

$$\begin{aligned} \tilde{X}^\mu &= x^\mu + \sqrt{2}\,\theta\chi^\mu \\ \tilde{Y}^\mu &= \tilde{X}^\mu - x_0^\mu \end{aligned} \tag{4.55}$$

with the same super conventions as before. The gluon vertex operator becomes (compare eq.(4.44))

$$V_{\mathrm{glu}}^A[k,\varepsilon,a] = T^a\int_0^T d\tau d\theta\,\varepsilon\cdot D\tilde{X}\exp[ik\cdot\tilde{X}] \tag{4.56}$$

(with T^a in the adjoint representation). The appropriate worldline super propagator is

$$\hat{G}_{P,A}^\chi(\tau_1,\theta_1;\tau_2,\theta_2) \equiv G_B(\tau_1,\tau_2) + 2\theta_1\theta_2 G_{P,A}^\chi(\tau_1,\tau_2) \tag{4.57}$$

The super Wick contraction rule

$$\langle \tilde{Y}^\mu(\tau_1,\theta_1)\tilde{Y}^\nu(\tau_2,\theta_2)\rangle_{P,A} = -g^{\mu\nu}\hat{G}_{P,A}^\chi(\tau_1,\theta_1;\tau_2,\theta_2) \tag{4.58}$$

then correctly reproduces the component field expressions. It allows us to take over all the conveniences of the superfield formalism encountered before, and to write the one-particle-irreducible off-shell N – gluon Green's function in a way analogous to eq.(4.46),

$$\begin{aligned}\Gamma^{a_1\dots a_N}_{\text{1PI,glu}}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] &= -\frac{1}{4}(-ig)^N \lim_{C\to\infty} \mathrm{tr}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-CT}(4\pi T)^{-\frac{D}{2}} \\ &\quad \times \sum_{p=P,A}\sigma_p Z_p \Big\langle V^A_{\text{glu},1}\cdots V^A_{\text{glu},N}\Big\rangle_p \delta(\frac{\tau_N}{T})\prod_{i=1}^{N-1}\theta(\hat{\tau}_{i(i+1)})\end{aligned} \tag{4.59}$$

Here we have defined $\sigma_P = 1$, $\sigma_A = -1$. $Z_{A,P}$ are the fermionic determinant factors

$$Z_{A,P} \equiv \mathrm{Det}_{A,P}\Big[(\frac{d}{d\tau} - C)\delta_{\mu\nu}\Big] = \Big(\mathrm{Det}_{A,P}\Big[\frac{d}{d\tau} - C\Big]\Big)^D \tag{4.60}$$

Those can be easily calculated using the same basis of circular eigenfunctions of the derivative operator as was used in the computation of G_B, G_F above. The result is

$$\begin{aligned} Z_A &= (2\cosh[CT/2])^D \\ Z_P &= (2\sinh[CT/2])^D \end{aligned} \tag{4.61}$$

Note that we have already set $D = 4$ in the reordering factor, and the same will be done for $Z_{A,P}$. This corresponds to the choice of a certain dimensional reduction variant of dimensional regularization, the four-dimensional helicity scheme developed by Bern and Kosower [21] (compare chapter 2).

The super formalism allows us not only to do without an additional two-gluon vertex operator, but also to generalize the replacement rule eq.(2.15) to the gluon loop case. This means that one can, for finite C and a fixed choice of the fermionic boundary conditions, first perform the bosonic Wick contractions, then partially integrate away all $\ddot{G}_{Bij}$'s, and finally include the terms from the fermionic sector by replacing

$$\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \to \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} - 2^n G^\chi_{p\,i_1i_2}G^\chi_{p\,i_2i_3}\cdots G^\chi_{p\,i_ni_1} \tag{4.62}$$

The analysis of the $C \to \infty$ limit [21,64] shows, however, that in this limit all terms containing multiple products of fermionic cycles get suppressed. Moreover, of the terms containing precisely one cycle, only those survive for which the ordering of the indices follows the ordering of the external legs. For those, the C – dependence is isolated in a factor of

$$z_n(C) \equiv e^{-CT}\frac{1}{2}\sum_{p=P,A}\sigma_p Z_p\, G^\chi_{p\,i_1i_2}G^\chi_{p\,i_2i_3}\cdots G^\chi_{p\,i_ni_1} \tag{4.63}$$

In the limit this expression goes to a constant, namely

$$\lim_{C\to\infty} z_n(C) = \begin{cases} 2 \quad : \quad n=2 \\ \\ 1 \quad : \quad n>2, \quad \text{indices follow legs} \\ \\ 0 \quad : \quad n>2, \quad \text{all other orderings} \end{cases} \tag{4.64}$$

Here the sign in the second line refers to the descending ordering

$$\tau_{i_1} > \tau_{i_2} > \ldots > \tau_{i_n} \tag{4.65}$$

If the ordering of the indices follows the ordering of the legs, this can always be arranged for using the cyclicity and the antisymmetry of $G^{\chi}_{A,P}$. This then leads just to the gluonic cycle rule part of the Bern-Kosower rules, eq.(2.17). For the remaining purely bosonic terms the C – dependence is trivial, and gives a factor

$$\lim_{C\to\infty} e^{-CT}\frac{1}{2}(Z_A - Z_P) = 4 \tag{4.66}$$

The bosonic part alone will therefore yield four times the contribution of a real scalar in the loop. This corresponds just to the four degrees of freedom of the gluon. But as we know from field theory the ghost contribution to the amplitude is -2 times the contribution of a real scalar. Therefore it just subtracts the contribution of the two unphysical degrees of freedom of the gluon, and the whole ghost contribution can be taken into account simply by changing the above factor from 4 to 2. This explains the factor 2 which we had in the Bern-Kosower rules for the "type 1" contributions to the gluon loop.

4.7. Example: QCD Vacuum Polarization

With all this machinery in place, we can now easily generalize our results for the QED vacuum polarization tensors, eqs.(4.38) and (4.41), to the QCD case [64,94].

For the QCD gluon vacuum polarization we have to take the scalar, spinor, gluon, and ghost loops into account. For the scalar and spinor loops, the replacement of eqs.(4.22),(4.23) by their non-abelian counterparts eqs.(4.45),(4.46) has the sole effect that the amplitude gets multiplied by a color trace $\mathrm{tr}(T^{a_1}T^{a_2})$. The δ – function term contained in $\theta(\tau_1 - \tau_2 + \theta_1\theta_2)$ after the θ – integrations produces a term proportional to

$$\delta_{12} G_{F12}\, \mathrm{e}^{-k^2 G_{B12}}$$

however its τ – integral vanishes since $G_{F12}(0) = 0$.

For the gluon loop, the coordinate part together with the ghost loop give the same as a complex scalar. Again the terms produced by the δ – function drop out due to the antisymmetry of $G^{\chi}_{A,P}(\tau_1,\tau_2)$. The only nontrivial new contribution comes from the gluon spin in the loop. This one is a two-cycle, and according to the above is related to the scalar contribution by a replacement of

$$\dot{G}_{B12}\dot{G}_{B21} \to 4$$

If we assume the loop scalars and fermions to be massless and in the adjoint representation, the results can be combined into the following single parameter integral (compare eqs.(4.37), (4.38),(4.41)),

$$\begin{aligned}
\Pi^{\mu\nu}_{\rm adj}(k) &= \mathrm{tr}\,(T^{a_1}_{\rm adj}T^{a_2}_{\rm adj})\frac{g^2}{(4\pi)^{\frac{D}{2}}}\Bigl[k^\mu k^\nu - g^{\mu\nu}k^2\Bigr]\Gamma(2-\frac{D}{2})\int_0^1 du\Bigl[u(1-u)k^2\Bigr]^{\frac{D}{2}-2}\\
&\quad\times\Bigl[(\frac{N_s}{2}-N_f+1)(1-2u)^2+N_f-4\Bigr] \qquad (4.67)
\end{aligned}$$

Here N_s denotes the number of (real) scalars, N_f the number of Weyl fermions. It should be remembered that, in field theory terms, this result corresponds to a calculation using the background field method and Feynman gauge. It is nice to verify [66,64] that the second line vanishes for $N_s = 6$, $N_f = 4$, corresponding to the case of $N = 4$ Super Yang-Mills theory, which is a finite theory. Note that the amplitude then vanishes already at the integrand level.

4.8. N Photon / N Gluon Amplitudes

Before proceeding to higher numbers of external legs, let us introduce some notation to keep the formulas manageable. Writing out the exponential in the master formula eq.(1.18) for a fixed number N of photons, one obtains an integrand

$$\exp\Bigl\{\Bigr\}\,|_{\rm multi-linear} = (-i)^N P_N(\dot{G}_{Bij},\ddot{G}_{Bij})\exp\Bigl[\frac{1}{2}\sum_{i,j=1}^N G_{Bij}k_i\cdot k_j\Bigr] \qquad (4.68)$$

with a certain polynomial P_N depending on the various $\dot{G}_{Bij}$'s, $\ddot{G}_{Bij}$'s, as well as on the kinematic invariants. To be able to apply the Bern-Kosower rules, we need to remove all second derivatives $\ddot{G}_{Bij}$ appearing by suitable partial integrations in the variables $\tau_1,\dots,\tau_N$. This transforms P_N into another polynomial Q_N depending only on the $\dot{G}_{Bij}$'s alone:

$$P_N(\dot{G}_{Bij},\ddot{G}_{Bij})\,\mathrm{e}^{\frac{1}{2}\sum G_{Bij}k_i\cdot k_j} \quad \stackrel{\rm part.int.}{\longrightarrow} \quad Q_N(\dot{G}_{Bij})\,\mathrm{e}^{\frac{1}{2}\sum G_{Bij}k_i\cdot k_j} \qquad (4.69)$$

As a result of the partial integration procedure certain combinations of the kinematic invariants are going to appear, the "Lorentz cycles" Z_n,

$$\begin{aligned}
Z_2(ij) &\equiv \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i - \varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\\
Z_n(i_1i_2\dots i_n) &\equiv \mathrm{tr}\prod_{j=1}^n\Bigl[k_{i_j}\otimes\varepsilon_{i_j} - \varepsilon_{i_j}\otimes k_{i_j}\Bigr] \quad (n\geq 3) \qquad (4.70)
\end{aligned}$$

Those generalize the transversal projector which is familiar from the two-point case. (In the (abelian) effective action they would correspond to a $\mathrm{tr}(F^n)$.) We also introduce the notion of a "τ - cycle", which is a product of $\dot{G}_{Bij}$'s with the indices forming a closed chain,

$$\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots \dot G_{Bi_ni_1} \tag{4.71}$$

(It should be remembered from chapter 2 that an expression is considered a cycle already if it can be put into cycle form using the antisymmetry of $\dot G_B$.) With these notations, the two-point result is

$$Q_2 \;=\; Z_2(12)\dot G_{B12}\dot G_{B21} \tag{4.72}$$

In the three - point case one starts with

$$\begin{aligned} P_3 \;&=\; \dot G_{B1i}\varepsilon_1\cdot k_i\dot G_{B2j}\varepsilon_2\cdot k_j\dot G_{B3k}\varepsilon_3\cdot k_k \\ &\quad -\Bigl[\ddot G_{B12}\varepsilon_1\cdot\varepsilon_2\dot G_{B3i}\varepsilon_3\cdot k_i \;+\;(1\to 2\to 3)\;+\;(1\to 3\to 2)\Bigr] \end{aligned} \tag{4.73}$$

Here and in the following the dummy indices i,j,k should be summed over from 1 to N, and one has $\dot G_{Bii}=0$ by antisymmetry. Removing all the $\ddot G_{Bij}$'s by partial integrations one finds

$$\begin{aligned} Q_3 \;&=\; \dot G_{B1i}\varepsilon_1\cdot k_i\dot G_{B2j}\varepsilon_2\cdot k_j\dot G_{B3k}\varepsilon_3\cdot k_k \\ &\quad +\frac{1}{2}\Bigl\{\dot G_{B12}\varepsilon_1\cdot\varepsilon_2\Bigl[\dot G_{B3i}\varepsilon_3\cdot k_i(\dot G_{B1j}k_1\cdot k_j-\dot G_{B2j}k_2\cdot k_j) \\ &\quad +(\dot G_{B31}\varepsilon_3\cdot k_1-\dot G_{B32}\varepsilon_3\cdot k_2)\dot G_{B3j}k_3\cdot k_j\Bigr]+2\text{ permutations}\Bigr\} \\ &=\; Q_3^3+Q_3^2 \end{aligned} \tag{4.74}$$

where

$$\begin{aligned} Q_3^3 \;&=\; \dot G_{B12}\dot G_{B23}\dot G_{B31}Z_3(123) \\ Q_3^2 \;&=\; \dot G_{B12}\dot G_{B21}Z_2(12)\dot G_{B3i}\varepsilon_3\cdot k_i+(1\to 2\to 3)+(1\to 3\to 2) \end{aligned} \tag{4.75}$$

Here we have decomposed the result of the partial integration, Q_3, according to its "cycle content", which is indicated by the upper index. Q_3^3 contains a 3-cycle, while the terms in Q_3^2 have a 2-cycle. Note that a τ - cycle comes multiplied with the corresponding Lorentz cycle. This turns out to be true in general, and motivates the further definition of a "bi-cycle" as the product of the two,

$$\dot G(i_1i_2\dots i_n)\equiv \dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}Z_n(i_1i_2\dots i_n) \tag{4.76}$$

After all bi-cycles have been separated out in a given term, whatever remains is called the "tail" of the term, or "m-tail", where m denotes the number of left-over $\dot G_{Bij}$'s. In the case of Q_3^2 we have just a 1-tail $\dot G_{B3i}\varepsilon_3\cdot k_i$.

In the abelian case the 3-point amplitude must vanish according to Furry's theorem. In the present formalism this can be immediately seen by noting that the integrand is odd under the orientation-reversing transformation of variables $\tau_i = T - \tau_i'$, $i = 1,2,3$.

In the three-point case, Q_3 is still unique; all possible ways of performing the partial integrations lead to the same result. The same is not true any more in the four-point case, where the result of the partial integration procedure turns out to depend on the specific chain of partial integrations chosen. In [147] it was shown that this ambiguity can be fixed by requiring Q_N to have, like P_N, the full permutation symmetry in the external legs, and a particular algorithm for the partial integration was given which manifestly preserves this symmetry. This algorithm is explained in detail in appendix C, where we also explicitly write down the resulting polynomials Q_N up to the six-point case. For the four-point case the result can be written in the following form,

$$
\begin{aligned}
Q_4 &= Q_4^4 + Q_4^3 + Q_4^2 + Q_4^{22} \\
Q_4^4 &= \dot G_{B12}\dot G_{B23}\dot G_{B34}\dot G_{B41} Z_4(1234) + 2 \text{ permutations} \\
Q_4^3 &= \dot G_{B12}\dot G_{B23}\dot G_{B31} Z_3(123)\dot G_{B4i}\varepsilon_4\cdot k_i + 3 \text{ perm.} \\
Q_4^2 &= \dot G_{B12}\dot G_{B21} Z_2(12){\sum}' \Bigl\{\dot G_{B3i}\varepsilon_3\cdot k_i \dot G_{B4j}\varepsilon_4\cdot k_j \\
&\quad + \frac{1}{2}\dot G_{B34}\varepsilon_3\cdot\varepsilon_4\Bigl[\dot G_{B3i}k_3\cdot k_i - \dot G_{B4i}k_4\cdot k_i\Bigr]\Bigr\} + 5 \text{ perm.} \\
Q_4^{22} &= \dot G_{B12}\dot G_{B21} Z_2(12)\dot G_{B34}\dot G_{B43} Z_2(34) + 2 \text{ perm.}
\end{aligned}
\tag{4.77}
$$

Here the terms in the partially integrated integrand have already been grouped according to their cycle content. The $\sum'$ appearing in the two-tail of Q_4^2 means that in the summation over the dummy variables i,j the term with $i = 4, j = 3$ must be omitted, since for these values an additional two-cycle would be present in the tail. Thus our final representation for the four-photon amplitude in scalar QED is the following,

$$
\begin{aligned}
\Gamma_{\rm scal}[k_1,\varepsilon_1;\ldots;k_4,\varepsilon_4] &= \frac{e^4}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}e^{-m^2T} \\
&\quad\times\int_0^1 du_1du_2du_3du_4\; Q_4(\dot G_{Bij})\exp\Bigl\{\frac{T}{2}\sum_{i,j=1}^4 G_{Bij}k_i\cdot k_j\Bigr\}
\end{aligned}
\tag{4.78}
$$

Here we have already rescaled to the unit circle, $\tau_i = Tu_i, G_{Bij} =\mid u_i - u_j \mid -(u_i-u_j)^2$. Note that this is already the complete (off-shell) amplitude, with no need to add "crossed" terms. The summation over crossed diagrams which would have to be done in a standard field theory calculation here is implicit in the integration over the various ordered sectors.

As an unexpected bonus of the whole procedure it turns out that this decomposition according to cycle content coincides, for arbitrary N, with a decomposition into gauge invariant partial amplitudes. Every single one of the 16 terms contained in the decomposition of Q_4 is individually gauge invariant, i.e. it either vanishes or turns into a total derivative if the replacement $\varepsilon_i \to k_i$ is made for any of the external legs. For example, if in Q_4^3 we substitute k_4 for ε_4 (in the un-permuted term) we have the following total derivative at hand,

$$\partial_4\Big[\dot G_{B12}\dot G_{B23}\dot G_{B31}\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}\Big]$$

These total derivatives become more and more complicated with increasing lengths of the tails [147]. Note also that the partial integration has the effect of homogeneizing the integrand; every term in Q_N has N factors of $\dot G_{Bij}$ and N factors of external momentum. In the four-point case this has the additional advantage of making the UV finiteness of the photon-photon scattering amplitude manifest. As is well-known, in a Feynman diagram calculation this property would be seen only after adding up all diagrams. Similarly, in the present approach the initial parameter integral still contains spurious divergences, since P_4 has terms involving products of two $\ddot G_{Bij}$'s which lead to a logarithmically divergent $T-$ integral. After the partial integration the integrand is finite term by term so that there is no necessity any more for an UV regulator; as far as the QED case is concerned, we can set $D = 4$ in (4.78).

Applying the Bern-Kosower rules to the above integrand we can immediately obtain the corresponding parameter integrals for (off-shell) photon-photon scattering in spinor QED, as well as for (on-shell) gluon-gluon scattering in QCD. As was already mentioned, in the latter case the presence of color factors forces one to restrict the integrations to the standard ordered sector $1 \geq u_1 \geq u_2 \geq u_3 \geq u_4 = 0$, and to explicitly sum over all non-cyclic permutations.

The same partial integration algorithm can also be used for the fermion loop in the superfield formalism. In appendix D this will be used for a simple proof of the replacement rule (2.15).

Finally, what does one gain by the partial integration procedure in terms of the difficulty of the arising parameter integrals? The partial integration increases the total number of terms in the integrand, but decreases the number of independent integrals. But then one must take into account the fact that those fewer independent integrals have, on the average, more complicated Feynman numerators (this fact turned out to be of significance in the calculation of the five – gluon amplitude [148]). It is therefore difficult to say in general, and to confuse matters more we will see in section 9.6 that sometimes even taking a linear combination of both integrands can be useful.

4.9. Example: Gluon – Gluon Scattering

Let us now have a look at a somewhat more substantial calculation, namely the one-loop gluon – gluon scattering amplitude in massless QCD. While this amplitude still does not present a challenge by modern standards, and was obtained by Ellis and Sexton many years ago [149], the advantages of the following recalculation over a standard Feynman diagram calculation should be evident.

The calculation proceeds in two parts. The first step is concerned with the reduction of this amplitude to a collection of parameter integrals; the second one with their explicit calculation. As always one finds those integrals to be of the same type as the corresponding Feynman parameter integrals. At the 4 - point level those integrals are still fairly easy. A convenient method for their calculation is described in appendix E, following [109].

Gluon scattering amplitudes are usually calculated in a helicity basis. A gluon has only two different physical polarizations, which are chosen to be helicity eigenstates "+" and "−". In QCD those amplitudes are not all independent, since CP invariance implies that simultaneously flipping all helicities is equivalent to changing all momenta from ingoing to outgoing, and vice versa. Therefore in the 4 - point case there are only four amplitudes to consider, $A(++++)$,

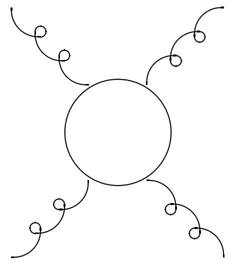


Figure 14: Gluon – gluon scattering amplitude.

$A(-+++)$, $A(--++)$, and $A(-+-+)$. As always in the non-abelian case we have to fix the ordering of the external legs, which we choose as the standard ordering (1234). In the end one must sum over all non-cyclic permutations.

Quite generally it turns out that the calculation of the N gluon or photon amplitudes is easiest for the all "+" (or all "−") cases, and most difficult for the completely "mixed" ones [14]. In the 4 - point case the calculation of $A(++++)$ is similar to the one of $A(-+++)$, while $A(--++)$ is similar to $A(-+-+)$. We will therefore restrict ourselves to the computation of $A(-+++)$ and $A(--++)$.

Let us start with the easier one, $A(-+++)$ (fig. 14). A given assignment of polarizations can be realized by many different choices of polarization vectors, and it is desirable to make this choice in such a way that the number of non-zero kinematic invariants $\varepsilon_i \cdot \varepsilon_j$, $\varepsilon_i \cdot k_j$ is minimized. A convenient way of finding such a set of polarization vectors for a given polarization assignment is provided by the spinor helicity technique (see, e.g., [111]), which makes use of the freedom to perform independent gauge transformations on all external legs. While this technique is already very useful in the corresponding field theory calculation, here its efficiency is further enhanced by the fact that there is no loop momentum, which reduces the number of kinematic invariants from the very beginning. Appropriate sets of polarization vectors have been given in [21,66]. For $A(-+++)$ they find that using a reference momentum k_4 for the first gluon and k_1 for the other ones makes all products of polarization vectors vanish,

$$\varepsilon_i \cdot \varepsilon_j = 0, \qquad i,j = 1,\ldots,4 \tag{4.79}$$

Moreover one has the following further relations,

$$\begin{aligned} &k_4 \cdot \varepsilon_1 = k_1 \cdot \varepsilon_2 = k_1 \cdot \varepsilon_3 = k_1 \cdot \varepsilon_4 = 0 \\ &k_3 \cdot \varepsilon_1 = -k_2 \cdot \varepsilon_1, \quad k_4 \cdot \varepsilon_2 = -k_3 \cdot \varepsilon_2, \quad k_4 \cdot \varepsilon_3 = -k_2 \cdot \varepsilon_3, \quad k_3 \cdot \varepsilon_4 = -k_2 \cdot \varepsilon_4 \end{aligned} \tag{4.80}$$

Those allow one to express all non-zero invariants in terms of $\varepsilon_1 \cdot k_3, \varepsilon_2 \cdot k_4, \varepsilon_3 \cdot k_4, \varepsilon_4 \cdot k_3$. Using

[14] It should be noted that $A(++++)$ does *not* describe a helicity conserving process, due to the convention used here that all momenta are ingoing.

them in the representation eq.(4.77) for Q_4 we find that they make most of the Lorentz cycles Z_n vanish; the surviving ones are

$$\begin{aligned} Z_3(234) &= 2\varepsilon_2 \cdot k_4 \varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_3 \\ Z_2(23) &= \varepsilon_2 \cdot k_4 \varepsilon_3 \cdot k_4 \\ Z_2(24) &= -\varepsilon_2 \cdot k_4 \varepsilon_4 \cdot k_3 \\ Z_2(34) &= \varepsilon_4 \cdot k_3 \varepsilon_3 \cdot k_4 \end{aligned} \tag{4.81}$$

This leads to the vanishing of all pure cycle terms,

$$Q_4^4 = Q_4^{22} = 0 \tag{4.82}$$

The surviving structures Q_4^3 and Q_4^2 yield

$$\begin{aligned} Q_4 &= C_{-+++}(\dot{G}_{B13} - \dot{G}_{B12}) \Big\{ 2\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B42} - \dot{G}_{B23}^2(\dot{G}_{B43} - \dot{G}_{B42}) \\ &\qquad + \dot{G}_{B24}^2(\dot{G}_{B34} - \dot{G}_{B32}) - \dot{G}_{B34}^2(\dot{G}_{B24} - \dot{G}_{B23}) \Big\} \end{aligned} \tag{4.83}$$

where

$$C_{-+++} \equiv \varepsilon_1 \cdot k_3 \varepsilon_2 \cdot k_4 \varepsilon_3 \cdot k_4 \varepsilon_4 \cdot k_3 \tag{4.84}$$

According to step 4 of the Bern - Kosower rules we should now search for possible pinch terms. The relevant ϕ^3 diagrams at this order are shown in fig. 15.

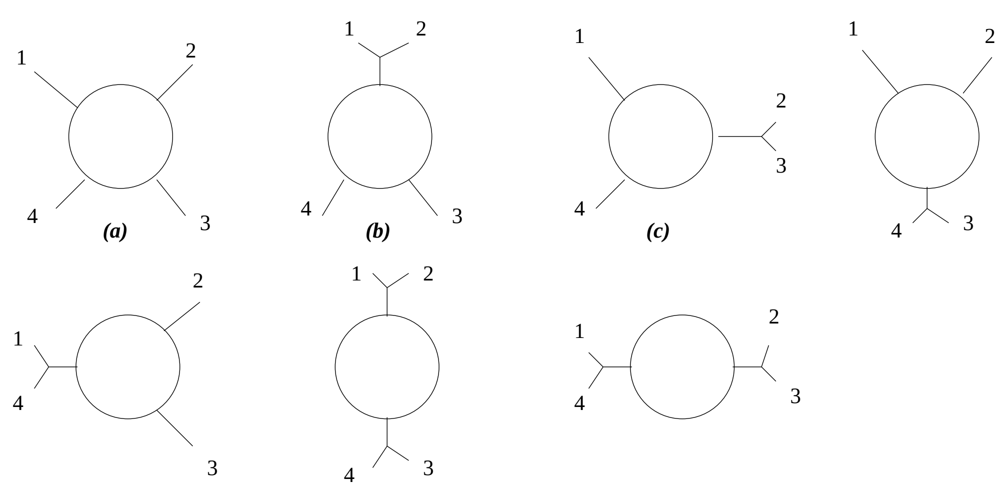


Figure 15: 4 - point ϕ^3 - diagrams

Considering diagram (b) we note that it is indeed a candidate for a pinch term, since Q_4 contains $\dot G_{B12}$ linearly. Step 5 of the Bern-Kosower rules tells us that to find its contribution we have to replace $\dot G_{B12}$ by $\frac{2}{(k_1+k_2)^2}$, and replace the variable u_2 by u_1 in the remainder of the expression. This transforms Q_4 into

$$
\begin{aligned}
P_b &= -2\frac{C_{-+++}}{(k_1+k_2)^2}\Bigl\{2\dot G_{B13}\dot G_{B34}\dot G_{B41} - \dot G^2_{B13}(\dot G_{B43}-\dot G_{B41}) + \dot G^2_{B14}(\dot G_{B34}-\dot G_{B31})\\
&\qquad -\dot G^2_{B34}(\dot G_{B14}-\dot G_{B13})\Bigr\}
\end{aligned}
\tag{4.85}
$$

Diagram (c) looks like another candidate, however its pinch contribution disappears since the first factor in Q_4 vanishes if u_3 is replaced by u_2. Similarly the pinch contributions of all remaining diagrams can be seen to be zero.

Thus for the scalar loop we have to calculate the following two parameter integrals,

$$
\begin{aligned}
D_a &= \Gamma(4-\frac{D}{2})\int_0^1 du_1\int_0^{u_1} du_2\int_0^{u_2} du_3\frac{P_a(u_1,u_2,u_3,u_4)}{\bigl[-\sum_{i<j=1}^4 G_{Bij}k_i\cdot k_j\bigr]^{4-\frac{D}{2}}}\\
D_b &= \Gamma(3-\frac{D}{2})\int_0^1 du_1\int_0^{u_1} du_3\frac{P_b(u_1,u_3,u_4)}{\bigl[-\sum_{i<j=1,3,4} G_{Bij}k_i\cdot k_j\bigr]^{3-\frac{D}{2}}}
\end{aligned}
\tag{4.86}
$$

(with P_a the Q_4 of eq.(4.83)). Both integrals turn out to be finite, so that one can set $D=4$. For their calculation we introduce Mandelstam variables s,t according to eq.(E.4) of appendix E, and write out the various G_{Bij}'s, $\dot G_{Bij}$'s for the standard ordering of the external legs, $u_1 > u_2 > u_3 > u_4 = 0$ (the usual rescaling to the unit circle has already been done). Then the first integral turns into

$$
D_a = -16C_{-+++}\int_0^1 du_1\int_0^{u_1} du_2\int_0^{u_2} du_3\frac{(u_2-u_3)^2u_3(1-u_2)}{\Bigl[s(u_2-u_3)(1-u_1)+t(u_1-u_2)u_3\Bigr]^2}
\tag{4.87}
$$

Beginning with u_1 all three integrations can be done elementarily, and one obtains

$$
D_a = -\frac{8}{3}\frac{C_{-+++}}{st}
\tag{4.88}
$$

Similarly the other diagram yields a

$$
D_b = -\frac{8}{3}\frac{C_{-+++}}{s^2}
\tag{4.89}
$$

To find the corresponding numerator polynomials for the spinor and gluon loops, we have to apply the cycle replacement rules eqs.(2.15),(2.17). For example, the first term inside the braces in eq.(4.83) is a 3-cycle, and will give additional terms both for the fermion and the gluon cases, and both for diagrams (a) and (b). However, adding up all terms generated by the application of the replacement rule, and writing them out in the u_i, one finds them to cancel out exactly. Thus for this particular helicity component there are no further integrals

to calculate, and the amplitudes for the scalar, fermion and gluon loop cases differ, apart from the group theory factor, only by the global factors counting the differences in statistics and degrees of freedom:

$$
\begin{aligned}
\Gamma_{\rm scal}^{a_1\cdots a_4}(-+++) &= -\frac{g^4}{12\pi^2}{\rm tr}(T_{\rm scal}^{a_1}\cdots T_{\rm scal}^{a_4})\frac{s+t}{s^2t}C_{-+++}\\
\Gamma_{\rm spin}^{a_1\cdots a_4}(-+++) &= \frac{g^4}{6\pi^2}{\rm tr}(T_{\rm spin}^{a_1}\cdots T_{\rm spin}^{a_4})\frac{s+t}{s^2t}C_{-+++}\\
\Gamma_{\rm glu}^{a_1\cdots a_4}(-+++) &= -\frac{g^4}{6\pi^2}{\rm tr}(T_{\rm adj}^{a_1}\cdots T_{\rm adj}^{a_4})\frac{s+t}{s^2t}C_{-+++}
\end{aligned}
\tag{4.90}
$$

Here the normalizations refer to a real scalar and to a Weyl fermion, and it should be remembered that the gluon loop contribution includes the contribution of its ghost. The factor C_{-+++} can, using the spinor helicity method, be expressed in terms of the Mandelstam variables up to a complex phase factor [111,66]. Remember that this is not yet the complete amplitude, but must still be summed over all non-cyclic permutations according to (2.19).

As in the case of the vacuum polarization, things become particularly simple if the scalars and fermions are also in the adjoint representation, which is the case in $N=4$ Super-Yang-Mills theory. Here one has

$$
\begin{aligned}
\Gamma_{\rm spin}(-+++) &= -2\Gamma_{\rm scal}(-+++)\\
\Gamma_{\rm glu}(-+++) &= 2\Gamma_{\rm scal}(-+++)
\end{aligned}
\tag{4.91}
$$

These simple relations hold only for $A(-+++)$ and $A(++++)$, and can be derived from the spacetime supersymmetry [66]. In contrast to a standard field theory calculation here they are visible already at the integrand level.

We proceed to the more substantial calculation of $A(--++)$. Again following [21,66] we choose reference momenta (k_4,k_4,k_1,k_1), which leads to the following relations:

$$
\begin{aligned}
\varepsilon_1\cdot\varepsilon_2 &= \varepsilon_1\cdot\varepsilon_3=\varepsilon_1\cdot\varepsilon_4=\varepsilon_2\cdot\varepsilon_4=\varepsilon_3\cdot\varepsilon_4=0\\
\varepsilon_1\cdot k_4 &= \varepsilon_2\cdot k_4=\varepsilon_3\cdot k_1=\varepsilon_4\cdot k_1=0\\
\varepsilon_1\cdot k_3 &= -\varepsilon_1\cdot k_2,\quad \varepsilon_2\cdot k_1=-\varepsilon_2\cdot k_3,\quad \varepsilon_3\cdot k_4=-\varepsilon_3\cdot k_2,\quad \varepsilon_4\cdot k_2=-\varepsilon_4\cdot k_3\\
\varepsilon_2\cdot\varepsilon_3 &= \frac{\varepsilon_2\cdot k_3\varepsilon_3\cdot k_2}{k_2\cdot k_3}
\end{aligned}
\tag{4.92}
$$

Using these relations in Q_4 one finds that this time Q_4^3 is vanishing. The others all do contribute, yielding

$$
\begin{aligned}
Q_4 &= C_{--++}\Bigl\{\dot G_{B12}^2\dot G_{B34}^2+\dot G_{B12}^2\Bigl[\dot G_{B23}\dot G_{B34}-\dot G_{B23}\dot G_{B24}-\dot G_{B24}\dot G_{B34}\Bigr]\\
&\quad +\dot G_{B34}^2\Bigl[\dot G_{B12}\dot G_{B23}-\dot G_{B12}\dot G_{B13}-\dot G_{B13}\dot G_{B23}\Bigr]+\dot G_{B12}\dot G_{B13}\dot G_{B24}\dot G_{B34}\\
&\quad +\dot G_{B13}\dot G_{B14}\dot G_{B23}\dot G_{B24}-\dot G_{B12}\dot G_{B14}\dot G_{B23}\dot G_{B34}\Bigr\}
\end{aligned}
\tag{4.93}
$$

with

$$C_{--++} \equiv \varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1 \varepsilon_3 \cdot k_2 \varepsilon_4 \cdot k_2 \tag{4.94}$$

For this helicity component all pinch terms turn out to vanish. For example, Q_4 contains four terms containing $\dot{G}_{B12}$ linearly, however they cancel in pairs once the replacement $u_2 \to u_1$ is made.

On the other hand, this time the replacement rules will have an effect. For example, for the spinor loop the first term in Q_4 has to be replaced by

$$\dot{G}^2_{B12}\dot{G}^2_{B34} \to (\dot{G}^2_{B12} - G^2_{F12})(\dot{G}^2_{B34} - G^2_{F34}) \tag{4.95}$$

The analogous replacement has to be made for the three $4-$ cycles appearing. For the gluon loop case the first term should instead be replaced by

$$\dot{G}^2_{B12}\dot{G}^2_{B34} \to \dot{G}^2_{B12}\dot{G}^2_{B34} - 4\dot{G}^2_{B12} - 4\dot{G}^2_{B34} \tag{4.96}$$

Of the three 4 - cycles only one has the ordering of the indices following the (standard) ordering of the external legs, and thus needs to be replaced by

$$\dot{G}_{B12}\dot{G}_{B14}\dot{G}_{B23}\dot{G}_{B34} \to \dot{G}_{B12}\dot{G}_{B14}\dot{G}_{B23}\dot{G}_{B34} - 8 \tag{4.97}$$

Writing out the results in the variables u_i, and then transforming to Feynman parameters according to eq.(E.3) of appendix E, one obtains the Feynman polynomials to be integrated. For the case of the gluon loop one finds, taking the global factor of 2 into account,

$$\begin{aligned} P_a \;=\; & 2C_{--++}\Big(8 - 12a_3 - 20a_1a_3 - 16a_2a_3 + 16a_1a_2a_3 + 16a_2^2a_3 - 4a_3^2 + 32a_1a_3^2 \\ & +64a_2a_3^2 - 48a_1a_2a_3^2 - 48a_2^2a_3^2 + 32a_3^3 - 32a_1a_3^3 - 48a_2a_3^3 - 16a_3^4\Big) \end{aligned} \tag{4.98}$$

For this helicity component the parameter integrals turn out to be divergent, so that their calculation is not elementary any more. In appendix E we explain a method [109] for the calculation of arbitrary on-shell massless four-point tensor integrals in dimensional regularization. For the numerator polynomial P_a this yields the following,

$$\begin{aligned} D_a \;=\; & 32\frac{C_{--++}}{st}\frac{\Gamma(1-\frac{\epsilon}{2})\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\epsilon)}\Big\{\frac{8}{\epsilon^2} + \frac{2\ln(s) + 2\ln(t) - \frac{11}{3}}{\epsilon} \\ & + \ln(s)\ln(t) - \frac{11}{6}\ln(t) - \frac{\pi^2}{2} + \frac{32}{9} + \mathrm{O}(\epsilon)\Big\} \end{aligned} \tag{4.99}$$

The double pole in this expansion is due to infrared divergences alone, while the simple pole comes from both infrared and ultraviolet divergences. The infrared divergences will ultimately cancel against contributions from the five-gluon tree amplitude, however the ultraviolet divergence must be removed by renormalization. The final result for the gluon loop contribution becomes

$$\Gamma_{\rm glu}^{a_1\cdots a_4}(--++) \;=\; \frac{g^4}{32\pi^2}\mathrm{tr}(T^{a_1}_{\rm adj}\cdots T^{a_4}_{\rm adj})(4\pi)^{-\frac{\epsilon}{2}}D_a^{\rm ren} \tag{4.100}$$

where

$$\begin{aligned} D_a^{\rm ren} \;=\; & 32\frac{C_{--++}}{st}\frac{\Gamma(1-\frac{\epsilon}{2})\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\epsilon)}\Biggl\{\frac{8}{\epsilon^2}+\frac{2\ln\Bigl(\frac{s}{\mu^2}\Bigr)+2\ln\Bigl(\frac{t}{\mu^2}\Bigr)-\frac{22}{3}}{\epsilon} \\ & +\ln\Bigl(\frac{s}{\mu^2}\Bigr)\ln\Bigl(\frac{t}{\mu^2}\Bigr)-\frac{11}{6}\ln\Bigl(\frac{t}{\mu^2}\Bigr)-\frac{\pi^2}{2}+\frac{32}{9}\Biggr\} \end{aligned} \tag{4.101}$$

and μ is the renormalization scale.

As usual we have worked in the Euclidean; the analytic continuation to physical momenta requires the use of the appropriate $i\epsilon$ prescription for the Mandelstam variables, $s \to s - i\epsilon$ etc. [15]

4.10. Boundary Terms and Gauge Invariance

So far we have completely disregarded possible boundary terms in the partial integration procedure. In the abelian case boundary terms are clearly absent, since all integrations are over the complete circle, and the integrand is written in terms of the worldline Green's functions, which have the appropriate periodicity properties. This is different in the non-abelian case, where boundary terms will generally appear. When using the string-derived rules for the calculation of the gluon scattering amplitudes those can still be ignored, since their contributions are automatically included by the application of the pinch rules. However, the validity of the original Bern-Kosower pinch rules is restricted to the on-shell case [16]. Therefore the boundary terms come into play if one wishes to apply the partial integration procedure to the calculation of the nonabelian effective action itself, or to the corresponding one-particle-irreducible off-shell vertex function $\Gamma_{\rm 1PI}[k_1,\ldots,\varepsilon_N]$. Let us therefore investigate their structure for the simplest case of a scalar loop (in this section we closely follow [150]).

Let us thus consider the standard low-energy expansion of the one-loop effective action in gauge theory, to be discussed at length in chapter 7 below. In scalar QED, the first non-trivial term in this expansion is proportional to the Maxwell term $F_{\mu\nu}F_{\mu\nu}$. Its coefficient is given by the zero-momentum limit of the vacuum polarization tensor, eq.(4.38). This term must also appear, with the same coefficient, in the scalar loop contribution to the QCD effective action. However by gauge invariance it must now involve the full non-Abelian field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \tag{4.102}$$

and thus be of the form

$$\mathrm{tr}\, F_{\mu\nu}F_{\mu\nu} = \mathrm{tr}\, F^0_{\mu\nu}F^0_{\mu\nu} + 2ig\mathrm{tr}\, F^0_{\mu\nu}[A_\mu, A_\nu] + \ldots \tag{4.103}$$

[15] Note that, due to our use of the metric $(-+++)$, our definition of the Mandelstam variables differs by a sign from the one used in [21,66].

[16] Some steps towards an extension of those rules to the off-shell case were taken in [64].

Here we have defined

$$F^0_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.104}$$

as the "abelian" part of the non-Abelian field strength tensor. Obviously, in the non-Abelian case the "bulk" integral will just produce the $F^0_{\mu\nu}F^0_{\mu\nu}$ - part. The additional terms involve the color commutator, and thus, in field theory terms, a quartic vertex. But Feynman diagrams involving quartic vertices have, compared to those with only cubic vertices, a smaller number of internal propagators, and thus of Feynman parameters. Since the boundary terms appearing in the worldline partial integration have fewer integrations but the same number of polarization vectors as the main term, they have the right structure to represent the missing color commutator terms, and a closer analysis of the effective action reveals that this is indeed their role. Here we will be satisfied with seeing how the second term in eq.(4.103) makes its appearance. Consider the three – point integrand before partial integration, the P_3 of eq.(4.73). If we take just the term

$$-\ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B3i}\varepsilon_3\cdot k_i \tag{4.105}$$

multiply by the three-point exponential, and partially integrate it in the variable τ_2, we find a boundary contribution

$$\int_0^T d\tau_1\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B3i}\varepsilon_3\cdot k_i \mathrm{e}^{\frac{1}{2}\sum G_{Bij}k_i\cdot k_j}\,|^{\tau_2=\tau_1}_{\tau_2=\tau_3} \quad = -\int_0^T d\tau_1\dot{G}_{B13}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B31}\varepsilon_3\cdot k_1\mathrm{e}^{G_{B13}k_1\cdot(k_2+k_3)} \tag{4.106}$$

($\tau_3 = 0$; only the lower boundary contributes, since $\dot{G}_{B12} = 0$ for $\tau_1 = \tau_2$). We note that the new integrand is, but for the Lorentz factors, identical with the one which we got in the two-point case after the partial integration, eq.(4.36). The momenta k_2 and k_3 now appear only in the combination $k_2 + k_3$, so that we may think of the vertex operators V_2 and V_3 as having merged to form a quartic vertex (fig. 16).

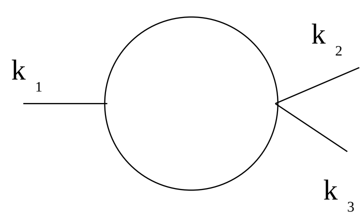


Figure 16: Structure of three-point boundary term

In our comparison with the effective action we thus should clearly identify leg number 1 with the A - field appearing in F^0, and indeed with this assignment the correct Lorentz structure ensues. What still needs to be seen is the color commutator. So far we have just a global color factor of tr $(T^{a_1}T^{a_2}T^{a_3})$, but we must remember that the full amplitude is only obtained after

summing over all non-cyclic permutations of the result reached with the standard ordering. In the three-point case we have already two non-equivalent orderings, and taking the other ordering into account one finds a second boundary contribution which is identical to the one above except that it has the reverse color trace. Both traces can then be combined to a $\mathrm{tr}\,(T^{a_1}[T^{a_2}, T^{a_3}])$.

We conclude that

1. Boundary terms always involve color commutators, and thus in the effective action picture contribute merely to the "covariantization" of the main term.

2. They lead to integrals which are already known from lower-point calculations.

The first fact may have been expected on general grounds; the second one is non-trivial, and has been verified only to low orders.

4.11. Relation to Feynman Diagrams

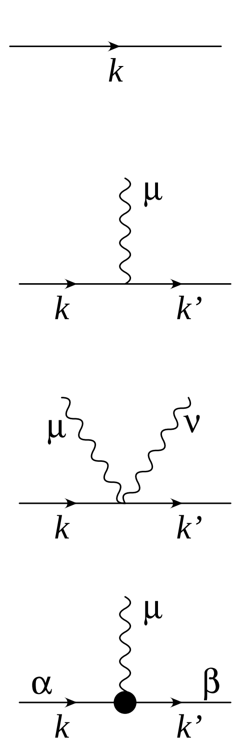
Finally, how do the integrand polynomials P_N relate to the ones encountered in an ordinary Feynman parameter calculation of the N – photon amplitude? For the scalar QED case the connection is still very direct [24]. Consider again the two Feynman diagrams for the scalar QED vacuum polarization, fig. 13. The first one is a tadpole diagram involving the seagull vertex. Now the result of the worldline calculation before partial integration, eq.(4.35), contained a $\ddot{G}_{Bij}$, and thus a $\delta(\tau_1 - \tau_2)$. This δ – function also creates a quartic vertex, and comparing the parameter integrals one finds that, not surprisingly, its contribution to the amplitude matches with the tadpole diagram.

This correspondence carries over to the N – point case, if one fixes the ordering of the external legs, and transforms from τ - to α - parameters according to eq.(4.15). The partially unintegrated Bern-Kosower integrand is thus obtained from the Feynman parameter integrand by a transformation of variables, and a certain regrouping of terms. This transformation has two effects. First, it allows one to combine into one expression an individual Feynman diagram and all the ones related to it by a permutation of the external states. Second, by regrouping the α – parameter expressions in terms of G_{Bij}, $\dot{G}_{Bij}$, $\ddot{G}_{Bij}$, which are functions well-adapted to the circle, the integrand is brought into a form suitable for partial integration, since now one needs, at least in the abelian case, not to worry about possible boundary terms.

In the spinor-loop case, comparison with the Feynman calculation is not quite so straightforward. The resulting parameter integrals obviously include those from the scalar loop, and thus contain contributions from diagrams including the seagull vertex. Clearly they cannot correspond to the parameter integrals obtained from the standard QED Feynman rules. It turns out that they correspond to a different break-up of those photon scattering amplitudes, a break-up according to a second-order formalism for fermions [151,152,153,154,24,64,155] (this holds true also for the more general theories considered in [101,103,156]).

The Feynman rules for (Euclidean) spinor QED in the second order formalism (see [155] and references therein) are, up to statistics and degrees of freedom, the ones for scalar QED with the addition of a third vertex (fig. 17).

The third vertex involves $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$ and corresponds to the $\psi^\mu F_{\mu\nu} \psi^\nu$ – term in the worldline Lagrangian $L_{\rm spin}$ (compare eq.(3.17)). For the details and for the non-abelian case



$$\frac{1}{k^2+m^2}$$

$$e\left(k+k'\right)_\mu$$

$$-2e^2 g_{\mu\nu}$$

$$e(\sigma_{\mu\nu})_{\beta\alpha}\left(k'-k\right)^\nu$$

Figure 17: Second order Feynman rules for spinor QED.

see [155]. There also an algorithm is given, based on the Gordon identity, which transforms the sum of Feynman (momentum) integrals resulting from the first order rules into the ones generated by the second order rules.

This explains the close relationship between scalar and spinor QED calculations in the worldline formalism, which we already noted before, and will encounter again at the multiloop level.

5. QED in a Constant External Field

An important role in quantum electrodynamics is played by processes involving constant external fields. An obvious physical reason is that in many cases a general field can be treated as a constant one to a good approximation. In QED this is expected to be the case if the variation of the field is small on the scale of the electron Compton wavelength. Mathematically, the constant field is distinguished by being one of the very few known field configurations for which the Dirac equation can be solved exactly, allowing one to obtain results which are nonperturbative in the field strength. For QED calculations in constant external fields it is possible and advantageous to take account of the field already at the level of the Feynman rules, i.e. to absorb it into the free electron propagator. Suitable formalisms have been developed many years ago [157,158,159,160]. However, beyond the simplest special cases they lead to extremely tedious and cumbersome calculations. As we will see in the present chapter, in the string-inspired formalism the inclusion of constant external fields requires only relatively minor modifications [96,97,98,99,92]. For this reason it has been extensively applied to constant field processes in QED in four [96,97,98,99,100,92,108,161,162,163,164] as well as in three dimensions [142].

5.1. Modified Worldline Green's Functions and Determinants

Similar to the absorption of a constant field into the electron propagator in standard field theory, in the worldline approach we would like to absorb the field into the basic worldline correlators. Let us thus assume that we have, in addition to the background field $A^\mu(x)$ we started with, a second one, $\bar{A}^\mu(x)$, with constant field strength tensor $\bar{F}_{\mu\nu}$. Using Fock–Schwinger gauge centered at x_0 [96] we may take $\bar{A}^\mu(x)$ to be of the form

$$\bar{A}_\mu(x) = \frac{1}{2} y^\nu \bar{F}_{\nu\mu} \tag{5.1}$$

The constant field contribution to the worldline Lagrangian (3.36) may then be written as

$$\Delta L_{\text{spin}} = \frac{1}{2} iey^\mu \bar{F}_{\mu\nu} \dot{y}^\nu - ie\psi^\mu \bar{F}_{\mu\nu} \psi^\nu \tag{5.2}$$

in components, or as

$$\Delta L_{\text{spin}} = -\frac{1}{2} ieY^\mu \bar{F}_{\mu\nu} DY^\nu \tag{5.3}$$

in the superfield formalism.

Since it is still quadratic in the worldline fields, we need not consider it as part of the interaction Lagrangian; we can absorb it into the free worldline propagators. This means that we need to replace the defining equations (1.15) and (4.18) for the worldline Green's functions by

$$2\langle \tau_1 \mid \left(\frac{d^2}{d\tau^2} - 2ie\bar{F} \frac{d}{d\tau} \right)^{-1} \mid \tau_2 \rangle \quad \equiv \quad \mathcal{G}_B(\tau_1, \tau_2) \tag{5.4}$$

$$2\langle \tau_1 \mid \left(\frac{d}{d\tau} - 2ie\bar{F} \right)^{-1} \mid \tau_2 \rangle \quad \equiv \quad \mathcal{G}_F(\tau_1, \tau_2) \tag{5.5}$$

These inverses are calculated in appendix B, with the result (deleting the "bar")

$$\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= \frac{T}{2(\mathcal{Z})^2}\Bigl(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} + i\mathcal{Z}\dot{G}_{B12} - 1\Bigr)\\
\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})}
\end{aligned} \tag{5.6}$$

where $\mathcal{Z} \equiv eFT$. These expressions should be understood as power series in the Lorentz matrix $\mathcal{Z}$ (note that eqs.(5.6) do not require the field strength tensor F to be invertible). Equivalent formulas have been given for the magnetic field case in [142], and for the general case in [99]. Note also that the generalized Green's functions are still translation invariant in τ, and thus functions of $\tau_1 - \tau_2$. By writing them in terms of the ordinary Green's function G_B we have avoided an explicit case distinction between $\tau_1 > \tau_2$ and $\tau_1 < \tau_2$ which would become necessary otherwise [99]. Note the symmetry properties

$$\mathcal{G}_B(\tau_1,\tau_2) = \mathcal{G}_B^T(\tau_2,\tau_1), \quad \dot{\mathcal{G}}_B(\tau_1,\tau_2) = -\dot{\mathcal{G}}_B^T(\tau_2,\tau_1), \quad \mathcal{G}_F(\tau_1,\tau_2) = -\mathcal{G}_F^T(\tau_2,\tau_1) \tag{5.7}$$

Those generalized Green's functions are, in general, nontrivial Lorentz matrices, so that the Wick contraction rules eq.(4.19) have to be replaced by

$$\begin{aligned}
\langle y^\mu(\tau_1)y^\nu(\tau_2)\rangle &= -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2)\\
\langle \psi^\mu(\tau_1)\psi^\nu(\tau_2)\rangle &= \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2)
\end{aligned} \tag{5.8}$$

We also need the generalizations of $\dot{G}_B, \ddot{G}_B$, which are (see appendix B)

$$\begin{aligned}
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &\equiv 2\langle\tau_1 \mid (\frac{d}{d\tau} - 2ieF)^{-1} \mid \tau_2\rangle = \frac{i}{\mathcal{Z}}\Bigl(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}} - 1\Bigr)\\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &\equiv 2\langle\tau_1 \mid \Bigl(\mathbb{1} - 2ieF(\frac{d}{d\tau})^{-1}\Bigr)^{-1} \mid \tau_2\rangle = 2\delta_{12} - \frac{2}{T}\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\mathrm{e}^{-i\mathcal{Z}\dot{G}_{B12}}
\end{aligned} \tag{5.9}$$

Let us also write down the first few terms in the expansion in $F_{\mu\nu}$ for all four functions,

$$\begin{aligned}
\mathcal{G}_{B12} &= G_{B12} - \frac{T}{6} - \frac{i}{3}\dot{G}_{B12}G_{B12}TeF + (\frac{T}{3}G_{B12}^2 - \frac{T^3}{90})(eF)^2 + O(F^3)\\
\dot{\mathcal{G}}_{B12} &= \dot{G}_{B12} + 2i(G_{B12} - \frac{T}{6})eF + \frac{2}{3}\dot{G}_{B12}G_{B12}T(eF)^2 + O(F^3)\\
\ddot{\mathcal{G}}_{B12} &= \ddot{G}_{B12} + 2i\dot{G}_{B12}eF - 4(G_{B12} - \frac{T}{6})(eF)^2 + O(F^3)\\
\mathcal{G}_{F12} &= G_{F12} - iG_{F12}\dot{G}_{B12}TeF + 2G_{F12}G_{B12}T(eF)^2 + O(F^3)
\end{aligned} \tag{5.10}$$

(here we used the identity $\dot{G}^2_{B12} = 1 - \frac{4}{T} G_{B12}$). To lowest order in this expansion the field dependent worldline Green's functions coincide, of course, with their vacuum counterparts.

Contrary to the vacuum case, in the constant field background one finds nonvanishing coincidence limits not only for $\mathcal{G}_B$, but also for $\dot{\mathcal{G}}_B$ and $\mathcal{G}_F$:

$$
\begin{aligned}
\mathcal{G}_B(\tau,\tau) &= \frac{T}{2(\mathcal{Z})^2}\Bigl(\mathcal{Z}\cot(\mathcal{Z}) - 1\Bigr) \\
\dot{\mathcal{G}}_B(\tau,\tau) &= i\cot(\mathcal{Z}) - \frac{i}{\mathcal{Z}} \\
\mathcal{G}_F(\tau,\tau) &= -i\tan(\mathcal{Z})
\end{aligned}
\tag{5.11}
$$

To correctly obtain this and other coincidence limits, one has to apply the rules

$$
\dot{G}_B(\tau,\tau) = 0, \quad \dot{G}^2_B(\tau,\tau) = 1 \tag{5.12}
$$

which follow from symmetry and continuity, respectively.

Again $\mathcal{G}_B$ and $\mathcal{G}_F$ may be assembled into a super propagator,

$$
\hat{\mathcal{G}}(\tau_1,\theta_1;\tau_2,\theta_2) \equiv \mathcal{G}_B(\tau_1,\tau_2) + \theta_1\theta_2\mathcal{G}_F(\tau_1,\tau_2) \tag{5.13}
$$

allowing one to generalize (4.30) to

$$
\langle Y^\mu(\tau_1,\theta_1) Y^\nu(\tau_2,\theta_2)\rangle = -\hat{\mathcal{G}}^{\mu\nu}(\tau_1,\theta_1;\tau_2,\theta_2) \tag{5.14}
$$

At first sight this definition would seem not to accommodate the non-vanishing coincidence limit of $\mathcal{G}_F$ (which can*not* be subtracted). Nevertheless, comparison with the component field formalism shows that the correct expressions are again reproduced if one takes coincidence limits *after* superderivatives. For instance, the correlator $\langle D_1 X(\tau_1,\theta_1) X(\tau_1,\theta_1)\rangle$ is evaluated by calculating

$$
\langle D_1 X(\tau_1,\theta_1) X(\tau_2,\theta_2)\rangle = \theta_1 \dot{\mathcal{G}}_{B12} - \theta_2 \mathcal{G}_{F12} \tag{5.15}
$$

and then setting $\tau_2 = \tau_1$.

This is almost all we need to know for computing one-loop photon scattering amplitudes, or the corresponding effective action, in a constant overall background field. The only further information required at the one–loop level is the change in the free path integral determinants due to the external field. As we will show in a moment, this change is ([96]; see also [38,165])

$$
(4\pi T)^{-\frac{D}{2}} \rightarrow (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] \qquad \text{(Scalar QED)} \tag{5.16}
$$

$$
(4\pi T)^{-\frac{D}{2}} \rightarrow (4\pi T)^{-\frac{D}{2}} \det{}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right] \qquad \text{(Spinor QED)} \tag{5.17}
$$

Since those determinants describe the vacuum amplitude in a constant field one finds them to be, of course, just the integrands of the well-known Euler-Heisenberg-Schwinger formulas.

5.2. Example: 1-Loop Euler-Heisenberg-Schwinger Lagrangians

Let us shortly retrace this calculation. In the scalar QED case, we have to replace

$$\int \mathcal{D}y \exp\Bigl[-\int_0^T d\tau \frac{1}{4}\dot{y}^2\Bigr] = \mathrm{Det'}_P^{-\frac{1}{2}}[-\frac{d^2}{d\tau^2}] = (4\pi T)^{-\frac{D}{2}} \tag{5.18}$$

by

$$\mathrm{Det'}_P^{-\frac{1}{2}}\Bigl[-\frac{d^2}{d\tau^2} + 2ieF\frac{d}{d\tau}\Bigr] = (4\pi T)^{-\frac{D}{2}}\mathrm{Det'}_P^{-\frac{1}{2}}\Bigl[\mathbb{1} - 2ieF(\frac{d}{d\tau})^{-1}\Bigr] \tag{5.19}$$

(as usual, the prime denotes the absence of the zero mode in a determinant). Application of the $ln\, det = tr\, ln$ identity yields [17]

$$\begin{aligned}
\mathrm{Det'}_P^{-\frac{1}{2}}\Bigl[\mathbb{1} - 2ieF(\frac{d}{d\tau})^{-1}\Bigr] &= \exp\Bigl\{\frac{1}{2}\sum_{n=1}^{\infty}\frac{(2ie)^n}{n}\mathrm{tr}[F^n]\mathrm{Tr}\Bigl[(\frac{d}{d\tau})^{-n}\Bigr]\Bigr\} \\
&= \exp\Bigl[-\frac{1}{2}\sum_{\substack{n=2 \\ n\ \mathrm{even}}}^{\infty}\frac{B_n}{n!n}(2ieT)^n\mathrm{tr}[F^n]\Bigr] \\
&= \det{}^{-\frac{1}{2}}\Bigl[\frac{\sin(eFT)}{eFT}\Bigr]
\end{aligned} \tag{5.20}$$

where the B_n are the Bernoulli numbers. In the second step eq.(B.9) was used. The analogous calculation for the Grassmann path integral yields a factor

$$\mathrm{Det}_A^{+\frac{1}{2}}\Bigl[\mathbb{1} - 2ieF(\frac{d}{d\tau})^{-1}\Bigr] = \det{}^{\frac{1}{2}}\Bigl[\cos(eFT)\Bigr] \tag{5.21}$$

For spinor QED we therefore find a total overall determinant factor of

$$(4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\Bigl[\frac{\tan(eFT)}{eFT}\Bigr] \tag{5.22}$$

Expressing these matrix determinants in terms of the two standard Lorentz invariants of the Maxwell field (see section 5.4 below) and continuing to Minkowski space, one obtains the well-known Schwinger proper-time representation of the (unrenormalized) Euler-Heisenberg-Schwinger Lagrangians [166,167,168,169],

$$\mathcal{L}_{\mathrm{scal}} = -\frac{1}{16\pi^2}\int_0^\infty \frac{ds}{s}\,\mathrm{e}^{-ism^2}\,\frac{e^2ab}{\sin(eas)\sinh(ebs)} \tag{5.23}$$

$$\mathcal{L}_{\mathrm{spin}} = \frac{1}{8\pi^2}\int_0^\infty \frac{ds}{s}\,\mathrm{e}^{-ism^2}\,\frac{e^2ab}{\tan(eas)\tanh(ebs)} \tag{5.24}$$

where $a^2 - b^2 \equiv \mathbf{B}^2 - \mathbf{E}^2$, $ab \equiv \mathbf{E}\cdot\mathbf{B}$.

[17] Note that, although the determinants considered here become formally identical with the ones appearing in (4.60) by $2ieF_{\mu\nu} \to C\delta_{\mu\nu}$, in the periodic case the results are not of the same form. The reason is that here the zero-mode had to be excluded from the determinant, while it needs to be included in the calculation of Z_P.

5.3. The N - Photon Amplitude in a Constant Field

Retracing our above calculation of the N - photon path integral with the external field included we arrive at the following generalization of eq.(1.18), representing the scalar QED N - photon scattering amplitude in a constant field [99,92]:

$$
\begin{aligned}
&\Gamma_{\rm scal}[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N] = (-ie)^N(2\pi)^D\delta(\sum k_i)\\
&\quad\times\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T}{\det}^{-\frac{1}{2}}\Bigl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Bigr]\prod_{i=1}^N\int_0^T d\tau_i\\
&\quad\times\exp\biggl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}k_i\cdot\mathcal{G}_{Bij}\cdot k_j - i\varepsilon_i\cdot\dot{\mathcal{G}}_{Bij}\cdot k_j+\frac{1}{2}\varepsilon_i\cdot\ddot{\mathcal{G}}_{Bij}\cdot\varepsilon_j\Bigr]\biggr\}\,|_{\rm multi-linear}
\end{aligned}
\tag{5.25}
$$

From this formula it is obvious that adding a constant Lorentz matrix to $\mathcal{G}_B$ will have no effect due to momentum conservation. As in the vacuum case, we can use this fact to get rid of the coincidence limit of $\mathcal{G}_B$, (5.11). Thus instead of $\mathcal{G}_B$ we will generally work with the equivalent Green's function $\bar{\mathcal{G}}_B$, defined by

$$
\bar{\mathcal{G}}_B(\tau_1,\tau_2)\equiv\mathcal{G}_B(\tau_1,\tau_2)-\mathcal{G}_B(\tau,\tau)=\frac{T}{2\mathcal{Z}}\Bigl(\frac{{\rm e}^{-i\dot{G}_{B12}\mathcal{Z}}-\cos(\mathcal{Z})}{\sin(\mathcal{Z})}+i\dot{G}_{B12}\Bigr) \tag{5.26}
$$

No such redefinition is possible for $\dot{\mathcal{G}}_B$ or $\mathcal{G}_F$.

The transition from scalar to spinor QED is done as in the vacuum case, again with only some minor modifications. The spinor QED integrand for a given number of photon legs N is obtained from the scalar QED integrand by the following generalization of the Bern-Kosower algorithm:

1. *Partial Integration:* After expanding out the exponential in the master formula (5.25), and taking the part linear in all $\varepsilon_1,\ldots,\varepsilon_N$, remove all second derivatives $\ddot{\mathcal{G}}_B$ appearing in the result by suitable partial integrations in $\tau_1,\ldots,\tau_N$.

2. *Replacement Rule:* Apply to the resulting new integrand the replacement rule (2.15) with $\dot{G}_B, G_F$ substituted by $\dot{\mathcal{G}}_B,\mathcal{G}_F$. Since the Green's functions $\mathcal{G}_B,\mathcal{G}_F$ are, in contrast to their vacuum counterparts, non-trivial matrices in the Lorentz indices, it must be mentioned here that the cycle property is defined solely in terms of the τ − indices, irrespectively of what happens to the Lorentz indices. For example, the expression

$$
\varepsilon_1\cdot\dot{\mathcal{G}}_{B12}\cdot k_2\,\varepsilon_2\cdot\dot{\mathcal{G}}_{B23}\cdot\varepsilon_3\,k_3\cdot\dot{\mathcal{G}}_{B31}\cdot k_1
$$

would have to be replaced by

$$
\varepsilon_1\cdot\dot{\mathcal{G}}_{B12}\cdot k_2\,\varepsilon_2\cdot\dot{\mathcal{G}}_{B23}\cdot\varepsilon_3\,k_3\cdot\dot{\mathcal{G}}_{B31}\cdot k_1\;-\varepsilon_1\cdot\mathcal{G}_{F12}\cdot k_2\,\varepsilon_2\cdot\mathcal{G}_{F23}\cdot\varepsilon_3\,k_3\cdot\mathcal{G}_{F31}\cdot k_1
$$

The only other difference to the vacuum case is due to the non-vanishing coincidence limits (5.11) of $\dot{\mathcal{G}}_B, \mathcal{G}_F$. Those lead to an extension of the "cycle replacement rule" to include one-cycles [92]:

$$\dot{\mathcal{G}}_B(\tau_i,\tau_i) \to \dot{\mathcal{G}}_B(\tau_i,\tau_i) - \mathcal{G}_F(\tau_i,\tau_i) \tag{5.27}$$

3. The scalar QED Euler-Heisenberg-Schwinger determinant factor must be replaced by its spinor QED equivalent,

$$\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] \to \det{}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right] \tag{5.28}$$

4. Multiply by the usual factor of -2 for statistics and degrees of freedom.

5.4. Explicit Representations of the Modified Worldline Green's Functions

For the result to be practically useful it will be necessary to write $\mathcal{G}_B, \mathcal{G}_F$ in more explicit form. This can be done by choosing some special Lorentz system, such as the one where **E** and **B** are both pointing along the z - direction, and working with the explicit matrix form of the worldline correlators, which becomes particularly simple in such a system. This approach turns out to be quite adequate for the case of a purely magnetic (or purely electric) field [92,100]. However, it is also possible to directly express all generalized worldline Green's functions in terms of Lorentz invariants, without specialization of the Lorentz frame. This procedure is not only more elegant but appears also to be more efficient computationally in the general case.

5.4.1. *Special Constant Fields*

1. *Magnetic field case:* With the B – field chosen along the z – axis, introduce matrices $g_\perp$ and $g_\parallel$ projecting on the x, y – and z, t – planes, so that

$$F = \begin{pmatrix} 0 & B & 0 & 0 \\ -B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, g_\perp \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, g_\parallel \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.29}$$

We also introduce $z = eBT$, and $\hat{F} = \frac{F}{B}$. With these notations, we can rewrite the determinant factors eqs.(5.16),(5.17) as

$$
\begin{aligned}
\det{}^{-\frac{1}{2}}\Bigl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Bigr] &= \frac{z}{\sinh(z)}\\
\det{}^{-\frac{1}{2}}\Bigl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\Bigr] &= \frac{z}{\tanh(z)}
\end{aligned}
\tag{5.30}
$$

The worldline correlators eqs.(5.6),(5.9),(5.26) specialize to

$$
\begin{aligned}
\bar{\mathcal{G}}_B(\tau_1,\tau_2) &= G_{B12}\,g_{\parallel} - \frac{T}{2}\frac{\Bigl(\cosh(z\dot{G}_{B12})-\cosh(z)\Bigr)}{z\sinh(z)}g_{\perp}\\
&\quad+\frac{T}{2z}\Bigl(\frac{\sinh(z\dot{G}_{B12})}{\sinh(z)}-\dot{G}_{B12}\Bigr)i\hat{F}\\
\dot{\mathcal{G}}_B(\tau_1,\tau_2) &= \dot{G}_{B12}\,g_{\parallel}+\frac{\sinh(z\dot{G}_{B12})}{\sinh(z)}g_{\perp}-\Bigl(\frac{\cosh(z\dot{G}_{B12})}{\sinh(z)}-\frac{1}{z}\Bigr)i\hat{F}\\
\ddot{\mathcal{G}}_B(\tau_1,\tau_2) &= \ddot{G}_{B12}\,g_{\parallel}+2\Bigl(\delta_{12}-\frac{z\cosh(z\dot{G}_{B12})}{T\sinh(z)}\Bigr)g_{\perp}+2\frac{z\sinh(z\dot{G}_{B12})}{T\sinh(z)}i\hat{F}\\
\mathcal{G}_F(\tau_1,\tau_2) &= G_{F12}\,g_{\parallel}+G_{F12}\frac{\cosh(z\dot{G}_{B12})}{\cosh(z)}g_{\perp}-G_{F12}\frac{\sinh(z\dot{G}_{B12})}{\cosh(z)}i\hat{F}
\end{aligned}
\tag{5.31}
$$

Note that from $\mathcal{G}_B$ we subtracted already its its coincidence limit, indicated by the "bar". Not removable are the coincidence limits for $\dot{\mathcal{G}}_B$ and $\mathcal{G}_F$,

$$
\begin{aligned}
\dot{\mathcal{G}}_B(\tau,\tau) &= -\Bigl(\coth(z)-\frac{1}{z}\Bigr)i\hat{F}\\
\mathcal{G}_F(\tau,\tau) &= -\tanh(z)i\hat{F}
\end{aligned}
\tag{5.32}
$$

2. *Crossed field case:* In a "crossed field", defined by $\mathbf{E}\perp\mathbf{B}, E=B$, both invariants B^2-E^2 and $\mathbf{E}\cdot\mathbf{B}$ vanish. For such a field $F^3=0$, so that the power series (5.6) break off after their quadratic terms. The worldline correlators thus get truncated to those terms which were given in (5.10). The determinant factors are trivial,

$$
\det{}^{-\frac{1}{2}}\Bigl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Bigr] = \det{}^{-\frac{1}{2}}\Bigl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\Bigr]=1 \tag{5.33}
$$

The importance of this case lies in the fact that a general constant field can be well-approximated by a crossed field at sufficiently high energies (see, e.g., [170]).

5.4.2. *Lorentz Covariant Decomposition for a General Field*

Defining the Maxwell invariants

$$
\begin{aligned}
f &\equiv \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}(B^2 - E^2) \\
g &\equiv \frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} = i\mathbf{E}\cdot\mathbf{B}
\end{aligned}
\tag{5.34}
$$

we have the relations

$$F^2 + \tilde{F}^2 = -2f\mathbb{1} \tag{5.35}$$

$$F\tilde{F} = -g\mathbb{1} \tag{5.36}$$

Define

$$F_\pm \equiv \frac{N_\pm^2 F - N_+ N_- \tilde{F}}{N_\pm^2 - N_\mp^2} \tag{5.37}$$

$$N_\pm \equiv n_+ \pm n_- \tag{5.38}$$

$$n_\pm \equiv \sqrt{\frac{f \pm g}{2}} \tag{5.39}$$

Then one has

$$F = F_+ + F_- \tag{5.40}$$

$$F^2 F_\pm = -N_\pm^2 F_\pm \tag{5.41}$$

$$F_+ F_- = 0 \tag{5.42}$$

With the help of these relations one easily derives the following formulas,

$$
\begin{aligned}
f_{\text{even}}(F) &= f_{\text{even}}(iN_+)\frac{F_+^2}{(iN_+)^2} + f_{\text{even}}(iN_-)\frac{F_-^2}{(iN_-)^2} \\
&= \frac{1}{N_+^2 - N_-^2}\Big\{-f_{\text{even}}(iN_+)[N_-^2\mathbb{1} + F^2] + f_{\text{even}}(iN_-)[N_+^2\mathbb{1} + F^2]\Big\} \\
f_{\text{odd}}(F) &= f_{\text{odd}}(iN_+)\frac{F_+}{iN_+} + f_{\text{odd}}(iN_-)\frac{F_-}{iN_-} \\
&= \frac{i}{N_+^2 - N_-^2}\Big\{[N_- f_{\text{odd}}(iN_-) - N_+ f_{\text{odd}}(iN_+)]F \\
&\qquad + [N_- f_{\text{odd}}(iN_+) - N_+ f_{\text{odd}}(iN_-)]\tilde{F}\Big\}
\end{aligned}
\tag{5.43}
$$

where $f_{\rm even}$ ($f_{\rm odd}$) are arbitrary even (odd) functions in the field strength matrix regular at $F=0$,

$$f_{\rm even}(F) = \sum_{n=0}^{\infty} c_{2n} F^{2n}, \qquad f_{\rm odd}(F) = \sum_{n=0}^{\infty} c_{2n+1} F^{2n+1} \tag{5.44}$$

Decomposing $\mathcal{G}_{B,F}$ into their even (odd) parts $\mathcal{S}_{B,F}$ ($\mathcal{A}_{B,F}$),

$$\mathcal{G}_{B,F} \;\; = \;\; \mathcal{S}_{B,F} + \mathcal{A}_{B,F} \tag{5.45}$$

and applying the above formulas we obtain the following matrix decompositions of $\mathcal{G}_B, \dot{\mathcal{G}}_B, \ddot{\mathcal{G}}_B, \mathcal{G}_F$,

$$\begin{aligned}
\mathcal{S}_{B12} &= \frac{T}{2}\Big[\frac{A^+_{B12}}{z_+}\hat{\mathcal{Z}}^2_+ + \frac{A^-_{B12}}{z_-}\hat{\mathcal{Z}}^2_-\Big] \\
&= \frac{T}{2(z_+^2 - z_-^2)}\Big\{\Big[\frac{z_-^2}{z_+}A^+_{B12} - \frac{z_+^2}{z_-}A^-_{B12}\Big]\mathbb{1} + \Big[\frac{A^+_{B12}}{z_+} - \frac{A^-_{B12}}{z_-}\Big]\mathcal{Z}^2\Big\} \\
\mathcal{A}_{B12} &= \frac{iT}{2}\Big[(S^+_{B12} - \dot{G}_{B12})\frac{\hat{\mathcal{Z}}_+}{z_+} + (S^-_{B12} - \dot{G}_{B12})\frac{\hat{\mathcal{Z}}_-}{z_-}\Big] \\
&= \frac{iT}{2(z_+^2 - z_-^2)}\Big\{[S^+_{B12} - S^-_{B12}]\mathcal{Z} + \Big[\frac{z_+}{z_-}(S^-_{B12} - \dot{G}_{B12}) - \frac{z_-}{z_+}(S^+_{B12} - \dot{G}_{B12})\Big]\tilde{\mathcal{Z}}\Big\} \\
\dot{\mathcal{S}}_{B12} &= -S^+_{B12}\hat{\mathcal{Z}}^2_+ - S^-_{B12}\hat{\mathcal{Z}}^2_- \\
&= \frac{1}{z_+^2 - z_-^2}\Big\{[z_+^2 S^-_{B12} - z_-^2 S^+_{B12}]\mathbb{1} + [S^-_{B12} - S^+_{B12}]\mathcal{Z}^2\Big\} \\
\dot{\mathcal{A}}_{B12} &= -i\Big[A^-_{B12}\hat{\mathcal{Z}}_- + A^+_{B12}\hat{\mathcal{Z}}_+\Big] \\
&= \frac{i}{z_+^2 - z_-^2}\Big\{[z_- A^-_{B12} - z_+ A^+_{B12}]\mathcal{Z} + [z_- A^+_{B12} - z_+ A^-_{B12}]\tilde{\mathcal{Z}}\Big\} \\
\ddot{\mathcal{S}}_{B12} &= \ddot{G}_{B12}\mathbb{1} + \frac{2}{T}\Big[z_+ A^+_{B12}\hat{\mathcal{Z}}^2_+ + z_- A^-_{B12}\hat{\mathcal{Z}}^2_-\Big] \\
&= \ddot{G}_{B12}\mathbb{1} + \frac{2}{T(z_+^2 - z_-^2)}\Big\{\Big[z_-^2 z_+ A^+_{B12} - z_+^2 z_- A^-_{B12}\Big]\mathbb{1} + \Big[z_+ A^+_{B12} - z_- A^-_{B12}\Big]\mathcal{Z}^2\Big\} \\
\ddot{\mathcal{A}}_{B12} &= \frac{2i}{T}\Big[z_+ S^+_{B12}\hat{\mathcal{Z}}_+ + z_- S^-_{B12}\hat{\mathcal{Z}}_-\Big] \\
&= \frac{2i}{T(z_+^2 - z_-^2)}\Big\{[z_+^2 S^+_{B12} - z_-^2 S^-_{B12}]\mathcal{Z} + z_+ z_-\Big[S^-_{B12} - S^+_{B12}\Big]\tilde{\mathcal{Z}}\Big\} \\
\mathcal{S}_{F12} &= -S^+_{F12}\hat{\mathcal{Z}}^2_+ - S^-_{F12}\hat{\mathcal{Z}}^2_- \\
&= \frac{1}{z_+^2 - z_-^2}\Big\{[z_+^2 S^-_{F12} - z_-^2 S^+_{F12}]\mathbb{1} + [S^-_{F12} - S^+_{F12}]\mathcal{Z}^2\Big\} \\
\mathcal{A}_{F12} &= -i\Big[A^-_{F12}\hat{\mathcal{Z}}_- + A^+_{F12}\hat{\mathcal{Z}}_+\Big] \\
&= \frac{i}{z_+^2 - z_-^2}\Big\{[z_- A^-_{F12} - z_+ A^+_{F12}]\mathcal{Z} + [z_- A^+_{F12} - z_+ A^-_{F12}]\tilde{\mathcal{Z}}\Big\}
\end{aligned} \tag{5.46}$$

Here we have further introduced

$$z_\pm \equiv eN_\pm T, \quad \tilde{\mathcal{Z}} \equiv eT\tilde{F}, \quad \mathcal{Z}_\pm \equiv eTF_\pm = \frac{z_\pm^2 \mathcal{Z} - z_+ z_- \tilde{\mathcal{Z}}}{z_\pm^2 - z_\mp^2}, \quad \hat{\mathcal{Z}}_\pm \equiv \frac{\mathcal{Z}_\pm}{z_\pm} \tag{5.47}$$

Note that $\mathcal{Z}\tilde{\mathcal{Z}} = -z_+ z_- \mathbb{1}$, $\hat{\mathcal{Z}}_\pm^3 = -\hat{\mathcal{Z}}_\pm$. The scalar, dimensionless coefficient functions appearing in these formulas are given by

$$\begin{aligned}
S_{B12}^{\pm} &= \frac{\sinh(z_\pm \dot{G}_{B12})}{\sinh(z_\pm)} \\
A_{B12}^{\pm} &= \frac{\cosh(z_\pm \dot{G}_{B12})}{\sinh(z_\pm)} - \frac{1}{z_\pm} \\
S_{F12}^{\pm} &= G_{F12}\frac{\cosh(z_\pm \dot{G}_{B12})}{\cosh(z_\pm)} \\
A_{F12}^{\pm} &= G_{F12}\frac{\sinh(z_\pm \dot{G}_{B12})}{\cosh(z_\pm)}
\end{aligned} \tag{5.48}$$

Note that $S_{B/F12}^{\pm}$ ($A_{B/F12}^{\pm}$) are odd (even) in $\tau_1 - \tau_2$. Thus the non-vanishing coincidence limits are in $A_{B,F}^{\pm}$,

$$\begin{aligned}
A_{Bii}^{\pm} &= \coth(z_\pm) - \frac{1}{z_\pm} \\
A_{Fii}^{\pm} &= \tanh(z_\pm)
\end{aligned} \tag{5.49}$$

In the string-inspired formalism, the functions (5.48) are the basic building blocks of parameter integrals for processes involving constant fields. Let us also write down the first few terms of the weak field expansions of these functions,

$$\begin{aligned}
S_{B12}^{\pm} &= \dot{G}_{B12}\Big[1 - \frac{2}{3}\frac{G_{B12}}{T}z_\pm^2 + \Big(\frac{2}{45}\frac{G_{B12}}{T} + \frac{2}{15}\frac{G_{B12}^2}{T^2}\Big)z_\pm^4 + \mathrm{O}(z_\pm^6)\Big] \\
A_{B12}^{\pm} &= \Big(\frac{1}{3} - 2\frac{G_{B12}}{T}\Big)z_\pm + \Big(-\frac{1}{45} + \frac{2}{3}\frac{G_{B12}^2}{T^2}\Big)z_\pm^3 + \mathrm{O}(z_\pm^5) \\
S_{F12}^{\pm} &= G_{F12}\Big[1 - 2\frac{G_{B12}}{T}z_\pm^2 + \frac{2}{3}\Big(\frac{G_{B12}}{T} + \frac{G_{B12}^2}{T^2}\Big)z_\pm^4 + \mathrm{O}(z_\pm^6)\Big] \\
A_{F12}^{\pm} &= G_{F12}\dot{G}_{B12}\Big[z_\pm - \Big(\frac{1}{3} + \frac{2}{3}\frac{G_{B12}}{T}\Big)z_\pm^3 + \mathrm{O}(z_\pm^5)\Big]
\end{aligned} \tag{5.50}$$

In the same way one finds for the determinant factors (5.16),(5.17)

$$
\begin{aligned}
\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right] &= \frac{z_+ z_-}{\sinh(z_+)\sinh(z_-)}, \\
\det{}^{-\frac{1}{2}}\left[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\right] &= \frac{z_+ z_-}{\tanh(z_+)\tanh(z_-)}
\end{aligned}
\tag{5.51}
$$

Using the above formulas we can obtain explicit results in a Lorentz covariant way. Nevertheless, it will be useful to write down these formulas also for the Lorentz system where $\mathbf{E}$ and $\mathbf{B}$ are both pointing along the positive z - axis, $\mathbf{E} = (0,0,E), \mathbf{B} = (0,0,B)$. (For this to be possible we have to assume that $\mathbf{E}\cdot\mathbf{B} > 0$.) In this Lorentz system $g = iEB$, so that

$$
n_\pm = \frac{1}{2}(B \pm iE), N_+ = B, N_- = iE, F_+ = Br_\perp, F_- = iEr_\parallel, \tag{5.52}
$$

where

$$
r_\perp \equiv \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad r_\parallel \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{5.53}
$$

Using those and the projectors $g_\perp, g\|$ introduced in (5.29) the matrix decompositions (5.46) can be rewritten as follows,

$$
\begin{aligned}
\mathcal{S}^{\mu\nu}_{B12} &= -\frac{T}{2}\sum_{\alpha=\perp,\parallel}\frac{A^\alpha_{B12}}{z_\alpha}\, g^{\mu\nu}_\alpha \\
\mathcal{A}^{\mu\nu}_{B12} &= \frac{iT}{2}\sum_{\alpha=\perp,\parallel}\frac{S^\alpha_{B12}-\dot{G}_{B12}}{z_\alpha}\, r^{\mu\nu}_\alpha \\
\dot{\mathcal{S}}^{\mu\nu}_{B12} &= \sum_{\alpha=\perp,\parallel} S^\alpha_{B12}\, g^{\mu\nu}_\alpha \\
\dot{\mathcal{A}}^{\mu\nu}_{B12} &= -i\sum_{\alpha=\perp,\parallel} A^\alpha_{B12}\, r^{\mu\nu}_\alpha \\
\ddot{\mathcal{S}}^{\mu\nu}_{B12} &= \ddot{G}_{B12} g^{\mu\nu} - \frac{2}{T}\sum_{\alpha=\perp,\parallel} z_\alpha A^\alpha_{B12}\, g^{\mu\nu}_\alpha \\
\ddot{\mathcal{A}}^{\mu\nu}_{B12} &= \frac{2i}{T}\sum_{\alpha=\perp,\parallel} z_\alpha S^\alpha_{B12}\, r^{\mu\nu}_\alpha
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{S}^{\mu\nu}_{F12} &= \sum_{\alpha=\perp,\parallel} S^{\alpha}_{F12}\, g^{\mu\nu}_{\alpha} \\
\mathcal{A}^{\mu\nu}_{F12} &= -i \sum_{\alpha=\perp,\parallel} A^{\alpha}_{F12}\, r^{\mu\nu}_{\alpha}
\end{aligned}
\tag{5.54}
$$

with $S/A^{\perp}_{B/F} \equiv S/A^{+}_{B/F}\,(z_+ = eBT \equiv z_\perp)$, $S/A^{\parallel}_{B/F} \equiv S/A^{-}_{B/F}\,(z_- = ieET \equiv z_\parallel)$.

5.5. Example: The Scalar/Spinor QED Vacuum Polarization Tensors in a Constant Field

We now apply this formalism to a calculation of the scalar and spinor QED vacuum polarization tensors in a general constant field. For the 2-point case the master formula (5.25) yields the following integrand,

$$
\exp\Bigl\{\dots\Bigr\}\,|_{\text{multi-linear}} \;=\Bigl[\varepsilon_1\cdot\ddot{\mathcal{G}}_{B12}\cdot\varepsilon_2 - \varepsilon_1\cdot\dot{\mathcal{G}}_{B1i}\cdot k_i\,\varepsilon_2\cdot\dot{\mathcal{G}}_{B2j}\cdot k_j\Bigr]\mathrm{e}^{k_1\cdot\bar{\mathcal{G}}_{B12}\cdot k_2}
\tag{5.55}
$$

where summation over $i,j=1,2$ is understood. Removing the second derivative in the first term by a partial integration in τ_1 this becomes

$$
\Bigl[-\varepsilon_1\cdot\dot{\mathcal{G}}_{B12}\cdot\varepsilon_2\,k_1\cdot\dot{\mathcal{G}}_{B1j}\cdot k_j - \varepsilon_1\cdot\dot{\mathcal{G}}_{B1i}\cdot k_i\,\varepsilon_2\cdot\dot{\mathcal{G}}_{B2j}\cdot k_j\Bigr]\mathrm{e}^{k_1\cdot\bar{\mathcal{G}}_{B12}\cdot k_2}
\tag{5.56}
$$

We apply the "cycle replacement rule" to this expression and use momentum conservation, $k\equiv k_1=-k_2$. The content of the brackets then turns into $\varepsilon_{1\mu}I^{\mu\nu}\varepsilon_{2\nu}$, where

$$
\begin{aligned}
I^{\mu\nu} \;=\;& \dot{\mathcal{G}}^{\mu\nu}_{B12}k\cdot\dot{\mathcal{G}}_{B12}\cdot k - \mathcal{G}^{\mu\nu}_{F12}k\cdot\mathcal{G}_{F12}\cdot k \\
&-\Bigl[\Bigl(\dot{\mathcal{G}}_{B11}-\mathcal{G}_{F11}-\dot{\mathcal{G}}_{B12}\Bigr)^{\mu\lambda}\Bigl(\dot{\mathcal{G}}_{B21}-\dot{\mathcal{G}}_{B22}+\mathcal{G}_{F22}\Bigr)^{\nu\kappa}+\mathcal{G}^{\mu\lambda}_{F12}\mathcal{G}^{\nu\kappa}_{F21}\Bigr]k^\kappa k^\lambda
\end{aligned}
\tag{5.57}
$$

Next we would like to use the fact that this integrand contains many terms which integrate to zero due to antisymmetry under the exchange $\tau_1\leftrightarrow\tau_2$. This we can do by decomposing $\mathcal{G}_B$ and $\mathcal{G}_F$ as in (5.45). First note that only the Lorentz even part of $\mathcal{G}_B$ contributes in the exponent,

$$
k_1\cdot\bar{\mathcal{G}}_{B12}\cdot k_2 \;=\; k_1\cdot(\mathcal{S}_{B12}-\mathcal{S}_{B11})\cdot k_2 \equiv -Tk\cdot\Phi_{12}\cdot k
\tag{5.58}
$$

$I^{\mu\nu}$ turns, after decomposing all factors of $\dot{\mathcal{G}}_B,\mathcal{G}_F$ as above, and deleting all τ - odd terms, into

$$
\begin{aligned}
I^{\mu\nu}_{\text{spin}} \;\equiv\;& \Bigl\{\Bigl(\dot{\mathcal{S}}^{\mu\nu}_{B12}\dot{\mathcal{S}}^{\kappa\lambda}_{B12}-\dot{\mathcal{S}}^{\mu\lambda}_{B12}\dot{\mathcal{S}}^{\nu\kappa}_{B12}\Bigr)-\Bigl(\mathcal{S}^{\mu\nu}_{F12}\mathcal{S}^{\kappa\lambda}_{F12}-\mathcal{S}^{\mu\lambda}_{F12}\mathcal{S}^{\nu\kappa}_{F12}\Bigr) \\
&+\Bigl(\dot{\mathcal{A}}_{B12}-\dot{\mathcal{A}}_{B11}+\mathcal{A}_{F11}\Bigr)^{\mu\lambda}\Bigl(\dot{\mathcal{A}}_{B12}-\dot{\mathcal{A}}_{B22}+\mathcal{A}_{F22}\Bigr)^{\nu\kappa} \\
&-\mathcal{A}^{\mu\lambda}_{F12}\mathcal{A}^{\nu\kappa}_{F12}\Bigr\}k^\kappa k^\lambda
\end{aligned}
\tag{5.59}
$$

(here we used (5.7)).

In this way we obtain the following integral representations for the dimensionally regularized scalar/spinor QED vacuum polarization tensors [171],

$$
\begin{aligned}
\Pi^{\mu\nu}_{\rm scal}(k) &= -\frac{e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}{\rm det}^{-\frac{1}{2}}\biggl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\biggr]\int_0^1 du_1\, I^{\mu\nu}_{\rm scal}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k} \qquad (5.60)\\
\Pi^{\mu\nu}_{\rm spin}(k) &= 2\frac{e^2}{(4\pi)^{\frac{D}{2}}}\int_0^\infty \frac{dT}{T}T^{2-\frac{D}{2}}e^{-m^2T}{\rm det}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\biggr]\int_0^1 du_1\, I^{\mu\nu}_{\rm spin}\,{\rm e}^{-Tk\cdot\Phi_{12}\cdot k} \qquad (5.61)
\end{aligned}
$$

Here $I^{\mu\nu}_{\rm scal}$ is obtained simply by deleting, in eq. (5.59), all quantities carrying a subscript "F". As usual we have rescaled to the unit circle and set $u_2 = 0$.

Note that again the transversality of the vacuum polarization tensors is manifest at the integrand level, $k_\mu I^{\mu\nu}_{\rm scal/spin} = I^{\mu\nu}_{\rm scal/spin}k_\nu = 0$.

The constant field vacuum polarization tensors contain the UV divergences of the ordinary vacuum polarization tensors (4.38),(4.41), and thus require renormalization. As is usual in this context we perform the renormalization on-shell, i.e. we impose the following condition on the renormalized vacuum polarization tensor $\bar\Pi^{\mu\nu}(k)$ (see, e.g., [172]),

$$
\lim_{k^2\to 0}\lim_{F\to 0}\bar\Pi^{\mu\nu}(k) = 0 \qquad (5.62)
$$

Counterterms appropriate to this condition are easy to find from our above results for the ordinary vacuum polarization tensors. From the representations eqs. (4.38), (4.41) for these tensors it is obvious that we can implement (5.62) by subtracting those same expressions with the last factor ${\rm e}^{-G_{B12}k^2}$ deleted.

In this way we find for the renormalized vacuum polarization tensors

$$
\begin{aligned}
\bar\Pi^{\mu\nu}_{\rm scal}(k) &= \Pi^{\mu\nu}_{\rm scal}(k) + \frac{\alpha}{4\pi}(g^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^1 du_1 \dot G_{B12}^2\\
\bar\Pi^{\mu\nu}_{\rm spin}(k) &= \Pi^{\mu\nu}_{\rm spin}(k) - \frac{\alpha}{2\pi}(g^{\mu\nu}k^2 - k^\mu k^\nu)\int_0^\infty \frac{dT}{T}e^{-m^2T}\int_0^1 du_1 (\dot G_{B12}^2 - G_{F12}^2)
\end{aligned}
\qquad (5.63)
$$

The remaining u_1 - integral can be brought into a more standard form by a transformation of variables $v = \dot G_{B12} = 1 - 2u_1$.

Writing the integrands explicitly using the formulas (5.46) and continuing to Minkoswki space [18] we obtain our final result for these amplitudes [164],

[18]For the Maxwell invariants this means $f \to \mathcal{F}$, $g \to i\mathcal{G}$, $N_+ \to a$, $N_- \to ib$ (to be able to fix all signs we assume $\mathcal{G} \geq 0$). Note also that $r_\perp k \to \tilde k_\perp, r_\parallel k \to -i\tilde k_\parallel$.

$$
\begin{aligned}
\bar{\Pi}^{\mu\nu}_{\rm scal}(k) &= -\frac{\alpha}{4\pi}\int_0^\infty \frac{ds}{s}\, e^{-ism^2}\int_{-1}^1 \frac{dv}{2}\biggl\{\frac{z_+z_-}{\sinh(z_+)\sinh(z_-)}\\
&\times\exp\biggl[-i\frac{s}{2}\sum_{\alpha=+,-}\frac{A^\alpha_{B12}-A^\alpha_{B11}}{z_\alpha}\,k\cdot\hat{\mathcal{Z}}^2_\alpha\cdot k\biggr]\\
&\times\sum_{\alpha,\beta=+,-}\biggl(S^\alpha_{B12}S^\beta_{B12}\Bigl[(\hat{\mathcal{Z}}^2_\alpha)^{\mu\nu}k\cdot\hat{\mathcal{Z}}^2_\beta\cdot k-(\hat{\mathcal{Z}}^2_\alpha k)^\mu(\hat{\mathcal{Z}}^2_\beta k)^\nu\Bigr]\\
&\qquad -(A^\alpha_{B12}-A^\alpha_{B11})(A^\beta_{B12}-A^\beta_{B22})(\hat{\mathcal{Z}}_\alpha k)^\mu(\hat{\mathcal{Z}}_\beta k)^\nu\biggr)\\
&-(\eta^{\mu\nu}k^2-k^\mu k^\nu)v^2\biggr\} \qquad (5.64)\\
\bar{\Pi}^{\mu\nu}_{\rm spin}(k) &= \frac{\alpha}{2\pi}\int_0^\infty \frac{ds}{s}\, e^{-ism^2}\int_{-1}^1 \frac{dv}{2}\biggl\{\frac{z_+z_-}{\tanh(z_+)\tanh(z_-)}\\
&\times\exp\biggl[-i\frac{s}{2}\sum_{\alpha=+,-}\frac{A^\alpha_{B12}-A^\alpha_{B11}}{z_\alpha}\,k\cdot\hat{\mathcal{Z}}^2_\alpha\cdot k\biggr]\\
&\times\sum_{\alpha,\beta=+,-}\biggl(\Bigl[S^\alpha_{B12}S^\beta_{B12}-S^\alpha_{F12}S^\beta_{F12}\Bigr]\Bigl[(\hat{\mathcal{Z}}^2_\alpha)^{\mu\nu}k\cdot\hat{\mathcal{Z}}^2_\beta\cdot k-(\hat{\mathcal{Z}}^2_\alpha k)^\mu(\hat{\mathcal{Z}}^2_\beta k)^\nu\Bigr]\\
&-\Bigl[(A^\alpha_{B12}-A^\alpha_{B11}+A^\alpha_{F11})(A^\beta_{B12}-A^\beta_{B22}+A^\beta_{F22})-A^\alpha_{F12}A^\beta_{F12}\Bigr](\hat{\mathcal{Z}}_\alpha k)^\mu(\hat{\mathcal{Z}}_\beta k)^\nu\biggr)\\
&-(\eta^{\mu\nu}k^2-k^\mu k^\nu)(v^2-1)\biggr\} \qquad (5.65)
\end{aligned}
$$

where now

$$
\begin{aligned}
z_+ &= iesa\\
z_- &= -esb\\
\hat{\mathcal{Z}}_+ &= \frac{aF-b\tilde{F}}{a^2+b^2}, \qquad \hat{\mathcal{Z}}^2_+ = \frac{F^2-b^2\mathbb{1}}{a^2+b^2}\\
\hat{\mathcal{Z}}_- &= -i\frac{bF+a\tilde{F}}{a^2+b^2}, \quad \hat{\mathcal{Z}}^2_- = -\frac{F^2+a^2\mathbb{1}}{a^2+b^2}
\end{aligned}
\qquad (5.66)
$$

Here a,b denote the standard 'secular' invariants which we already introduced in eqs.(5.23),(5.24). In terms of the invariants $\mathcal{F},\mathcal{G}$ those read

$$
\begin{aligned}
a &\equiv \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}+\mathcal{F}}\\
b &\equiv \sqrt{\sqrt{\mathcal{F}^2+\mathcal{G}^2}-\mathcal{F}}
\end{aligned}
\qquad (5.67)
$$

($\mathcal{F}=\frac{1}{2}(B^2-E^2)$, $\quad\mathcal{G}=\mathbf{E}\cdot\mathbf{B}$).

For fermion QED, the constant field vacuum polarization tensor was obtained before by various authors [173,174,175,176]. For the sake of comparison with their results, let us also specialize

to the Lorentz system where $\mathbf{E} = (0,0,E)$ and $\mathbf{B} = (0,0,B)$. In this system $a = B, b = E$. Denoting

$$
\begin{aligned}
k_{\|} &= (k^0,0,0,k^3), \quad k_{\perp} = (0,k^1,k^2,0) \\
\tilde{k}_{\|} &= (k^3,0,0,k^0), \quad \tilde{k}_{\perp} = (0,k^2,-k^1,0)
\end{aligned}
\tag{5.68}
$$

our result can be written as follows,

$$
\begin{aligned}
\bar{\Pi}^{\mu\nu}_{\binom{\text{spin}}{\text{scal}}}(k) &= -\frac{\alpha}{4\pi}\begin{pmatrix}-2\\1\end{pmatrix}\int_0^\infty \frac{ds}{s}\int_{-1}^1 \frac{dv}{2}\Biggl\{\frac{zz'}{\sin(z)\sinh(z')}\begin{pmatrix}\cos(z)\cosh(z')\\1\end{pmatrix} \\
&\times \mathrm{e}^{-is\Phi_0}\sum_{\alpha,\beta=\perp,\|}\Bigl[s^{\alpha\beta}_{\binom{\text{spin}}{\text{scal}}}(\eta^{\mu\nu}_{\alpha}k^2_{\beta} - k^{\mu}_{\alpha}k^{\nu}_{\beta}) + a^{\alpha\beta}_{\binom{\text{spin}}{\text{scal}}}\tilde{k}^{\mu}_{\alpha}\tilde{k}^{\nu}_{\beta}\Bigr] \\
&-\mathrm{e}^{-ism^2}(\eta^{\mu\nu}k^2 - k^{\mu}k^{\nu})\begin{pmatrix}v^2-1\\v^2\end{pmatrix}\Biggr\}
\end{aligned}
\tag{5.69}
$$

where $z = eBs, z' = eEs$, and

$$
\Phi_0 = m^2 + \frac{k^2_{\perp}}{2}\frac{\cos(zv)-\cos(z)}{z\sin(z)} - \frac{k^2_{\|}}{2}\frac{\cosh(z'v)-\cosh(z')}{z'\sinh(z')}
\tag{5.70}
$$

$$
\begin{aligned}
s^{\perp\perp}_{\text{scal}} &= \frac{\sin^2(zv)}{\sin^2(z)} \\
s^{\perp\|,\|\perp}_{\text{scal}} &= \frac{\sin(zv)\sinh(z'v)}{\sin(z)\sinh(z')} \\
s^{\|\|}_{\text{scal}} &= \frac{\sinh^2(z'v)}{\sinh^2(z')} \\
a^{\perp\perp}_{\text{scal}} &= \Bigl(\frac{\cos(zv)-\cos(z)}{\sin(z)}\Bigr)^2 \\
a^{\perp\|,\|\perp}_{\text{scal}} &= -\frac{\cos(zv)-\cos(z)}{\sin(z)}\frac{\cosh(z'v)-\cosh(z')}{\sinh(z')} \\
a^{\|\|}_{\text{scal}} &= \Bigl(\frac{\cosh(z'v)-\cosh(z')}{\sinh(z')}\Bigr)^2
\end{aligned}
\tag{5.71}
$$

$$
s^{\perp\perp}_{\text{spin}} = \frac{\sin^2(zv)}{\sin^2(z)} - \frac{\cos^2(zv)}{\cos^2(z)}
$$

$$
\begin{aligned}
s_{\rm spin}^{\perp\|,\|\perp} &= \frac{\sin(zv)\sinh(z'v)}{\sin(z)\sinh(z')} - \frac{\cos(zv)\cosh(z'v)}{\cos(z)\cosh(z')}\\
s_{\rm spin}^{\|\|} &= \frac{\sinh^2(z'v)}{\sinh^2(z')} - \frac{\cosh^2(z'v)}{\cosh^2(z')}\\
a_{\rm spin}^{\perp\perp} &= \Bigl(\frac{\cos(zv)-\cos(z)}{\sin(z)} - \tan(z)\Bigr)^2 - \frac{\sin^2(zv)}{\cos^2(z)}\\
a_{\rm spin}^{\perp\|,\|\perp} &= -\Bigl(\frac{\cos(zv)-\cos(z)}{\sin(z)} - \tan(z)\Bigr)\Bigl(\frac{\cosh(z'v)-\cosh(z')}{\sinh(z')} + \tanh(z')\Bigr)\\
&\quad -\frac{\sin(zv)\sinh(z'v)}{\cos(z)\cosh(z')}\\
a_{\rm spin}^{\|\|} &= \Bigl(\frac{\cosh(z'v)-\cosh(z')}{\sinh(z')} + \tanh(z')\Bigr)^2 - \frac{\sinh^2(z'v)}{\cosh^2(z')}
\end{aligned}
\tag{5.72}
$$

In this form it can be easily identified with the field theory results of [159] (scalar QED) and [174,175] (fermion QED).

5.6. Example: Photon Splitting in a Constant Magnetic Field

Photon splitting in a constant magnetic field is a process of potential astrophysical interest. Its first exact calculation, valid for an arbitrary magnetic field strength and photon energies up to the pair creation threshold, was performed by Adler in 1971 [177]. This calculation amounts essentially to the calculation of the QED one-loop three-photon amplitude in a constant field. This amplitude is finite, so that one can set $D = 4$.

5.6.1. *Scalar QED*

To obtain the photon splitting amplitude for scalar QED, we have to use the correlators (5.31) for the Wick contraction of three photon vertex operators V_0 and $V_{1,2}$,

$$
V^A_{{\rm scal},i}[k_i,\varepsilon_i] = \int_0^T d\tau_i\,\varepsilon_i\cdot\dot x(\tau_i)\exp\Bigl[ik_i\cdot x(\tau_i)\Bigr]
$$

representing the incoming and the two outgoing photons.

The calculation is greatly simplified by the peculiar kinematics of this process. Energy–momentum conservation $k_0 + k_1 + k_2 = 0$ forces collinearity of all three four–momenta, so that, writing $k_0 \equiv k \equiv \omega n$,

$$
k_1 = -\frac{\omega_1}{\omega}k, k_2 = -\frac{\omega_2}{\omega}k;\; k^2 = k_1^2 = k_2^2 = k\cdot k_1 = k\cdot k_2 = k_1\cdot k_2 = 0. \tag{5.73}
$$

By a simple Lorentz invariance argument [177] one can assume $\mathbf{k}$ to be orthogonal to the magnetic field direction. Moreover, in [177] it was shown, using CP invariance together with an analysis of dispersive effects, that there is only one non-vanishing polarization case. This is the case where the magnetic vector $\hat{\mathbf{k}}\times\hat{\varepsilon_{\mathbf{0}}}$ of the incoming photon is parallel to the plane containing the external field and the direction of propagation $\hat{\mathbf{k}}$, and those of the outgoing

ones are both perpendicular to this plane. Taking the magnetic field in the z - direction, and choosing [19] $n = (1,0,0,-i)$, we can implement this case by taking $\varepsilon_0 = (0,1,0,0)$ and $\varepsilon_1 = \varepsilon_2 = (0,0,1,0)$. This leads to the further vanishing relations

$$\varepsilon_{1,2} \cdot \varepsilon_0 = \varepsilon_{1,2} \cdot k = \varepsilon_{1,2} \cdot F = 0 \tag{5.74}$$

which leave us with the only a small number of nonvanishing Wick contractions:

$$\begin{aligned}
\langle V^A_{\rm scal,0} V^A_{\rm scal,1} V^A_{\rm scal,2} \rangle &= \Bigl\langle \prod_{i=0}^{2} \int_0^T d\tau_i \, \varepsilon_i \cdot \dot{x}_i \exp\Bigl[i k_i \cdot x(\tau_i) \Bigr] \Bigr\rangle \\
&= \Bigl\langle \prod_{i=0}^{2} \int_0^T d\tau_i \, \varepsilon_i \cdot \dot{x}_i \exp\Bigl[i\, n \cdot \sum_{j=0}^{2} \bar{\omega}_j x_j \Bigr] \Bigr\rangle \\
&= -i \prod_{i=0}^{2} \int_0^T d\tau_i \exp\Bigl[\frac{1}{2} \sum_{i,j=0}^{2} \bar{\omega}_i \bar{\omega}_j n \cdot \bar{\mathcal{G}}_{Bij} \cdot n \Bigr] \varepsilon_1 \cdot \ddot{\mathcal{G}}_{B12} \cdot \varepsilon_2 \sum_{i=0}^{2} \bar{\omega}_i \varepsilon_0 \cdot \dot{\mathcal{G}}_{B0i} \cdot n
\end{aligned} \tag{5.75}$$

To keep the notation compact, we have defined $\bar{\omega}_0 = \omega, \bar{\omega}_{1,2} = -\omega_{1,2}$. A number of terms which vanish by the above relations have been omitted. For example, to see that the term involving

$$\langle \varepsilon_0 \cdot \dot{x}_0 \, \varepsilon_1 \cdot \dot{x}_1 \rangle = \varepsilon_0 \cdot \ddot{\mathcal{G}}_{B01} \cdot \varepsilon_1$$

vanishes, remember that $\ddot{\mathcal{G}}_B$ is a power series in the matrix $F_{\mu\nu}$. The first term in this expansion is proportional to the Lorentz identity and gives zero since $\varepsilon_0 \cdot \varepsilon_1 = 0$; all remaining ones give zero because $F \cdot \varepsilon_1 = 0$.

Performing the Lorentz contractions, and taking the determinant factor eq.(5.30) into account, one obtains the following parameter integral for the three-point amplitude:

$$\begin{aligned}
\Gamma_{\rm scal}[k_0,k_1,k_2] &= (-ie)^3 \int_0^\infty \frac{dT}{T} {\rm e}^{-m^2 T} (4\pi T)^{-2} \frac{z}{\sinh(z)} \langle V^A_{\rm scal,0} V^A_{\rm scal,1} V^A_{\rm scal,2} \rangle \\
&= i e^3 \int_0^\infty dT \, {\rm e}^{-m^2 T} (4\pi T)^{-2} \frac{z}{\sinh(z)} \int_0^T d\tau_1 \, d\tau_2 \, \ddot{G}_{B12} \sum_{i=0}^{2} \bar{\omega}_i \frac{\cosh(z \dot{G}_{B0i})}{\sinh(z)} \\
&\quad \times \exp\Bigl\{ -\frac{1}{2} \sum_{i,j=0}^{2} \bar{\omega}_i \bar{\omega}_j \Bigl[G_{Bij} + \frac{T}{2z} \frac{\cosh(z \dot{G}_{Bij})}{\sinh(z)} \Bigr] \Bigr\}
\end{aligned} \tag{5.76}$$

($z = eBT$). Translation invariance in τ has been used to set the position τ_0 of the incoming photon equal to T. Normalizing the amplitude according to eq.(25) in [177], the final result becomes

[19] Note that we are still using Euclidean conventions.

$$
\begin{aligned}
C_{\rm scal}[\omega,\omega_1,\omega_2,B] &= \frac{m^8}{8\omega\omega_1\omega_2}\int_0^\infty dT\,T\frac{\mathrm{e}^{-m^2T}}{z^2\sinh^2(z)}\int_0^T d\tau_1\,d\tau_2\,\ddot G_{B12}\\
&\quad\times\Bigl[\sum_{i=0}^2\bar\omega_i\cosh(z\dot G_{B0i})\Bigr]\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{T}{2z}\frac{\cosh(z\dot G_{Bij})}{\sinh(z)}\Bigr]\Bigr\}
\end{aligned}
\tag{5.77}
$$

($C_{\rm scal}$ corresponds to C_2 there).

5.6.2. Spinor QED

For the fermion loop case let us, for a change, use the superfield formalism rather than the cycle replacement rule. Using the superfield representation (4.29) of the photon vertex operator,

$$
V^A_{\rm spin}[k,\varepsilon]=\int_0^T d\tau d\theta\,\varepsilon\cdot DX\mathrm{e}^{ik\cdot X}
$$

we can write the result of the Wick contraction for the spinor loop in complete analogy to the scalar loop result eq.(5.75):

$$
\langle V^A_{{\rm spin},0}V^A_{{\rm spin},1}V^A_{{\rm spin},2}\rangle = i\prod_{i=0}^2\int_0^T d\tau_i\int d\theta_i\,\exp\Bigl[\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j n\cdot\bar{\hat{\mathcal{G}}}_{ij}\cdot n\Bigr]\,\varepsilon_1\cdot D_1D_2\hat{\mathcal{G}}_{12}\cdot\varepsilon_2\,\sum_{i=0}^2\bar\omega_i\varepsilon_0\cdot D_0\hat{\mathcal{G}}_{0i}\cdot n
\tag{5.78}
$$

Here $\hat{\mathcal{G}}$ denotes the constant field worldline super propagator, eq.(5.13). The Lorentz contractions are performed as before. The only difference is in the additional θ − integrations, which are easy to do. The final result becomes

$$
\begin{aligned}
&C_{\rm spin}[\omega,\omega_1,\omega_2,B]=\frac{m^8}{4\omega\omega_1\omega_2}\int_0^\infty dT\,T\frac{\mathrm{e}^{-m^2T}}{z^2\sinh(z)}\\
&\times\int_0^T d\tau_1\,d\tau_2\,\exp\Bigl\{-\frac{1}{2}\sum_{i,j=0}^2\bar\omega_i\bar\omega_j\Bigl[G_{Bij}+\frac{T}{2z}\frac{\cosh(z\dot G_{Bij})}{\sinh(z)}\Bigr]\Bigr\}\\
&\times\biggl\{\Bigl[-\cosh(z)\ddot G_{B12}+\omega_1\omega_2\Bigl(\cosh(z)-\cosh(z\dot G_{B12})\Bigr)\Bigr]\\
&\times\Bigl[\frac{\omega}{\sinh(z)\cosh(z)}-\omega_1\frac{\cosh(z\dot G_{B01})}{\sinh(z)}-\omega_2\frac{\cosh(z\dot G_{B02})}{\sinh(z)}\Bigr]\\
&+\frac{\omega\omega_1\omega_2 G_{F12}}{\cosh(z)}\Bigl[\sinh(z\dot G_{B01})\Bigl(\cosh(z)-\cosh(z\dot G_{B02})\Bigr)-(1\leftrightarrow 2)\Bigr]\biggr\}
\end{aligned}
\tag{5.79}
$$

A numerical analysis of this three-parameter integral has shown [100,178] it to be in complete agreement with other known integral representations of this amplitude [177,160,179,180].

See [181] for a more extensive analysis, as well as for a discussion of the relevance of photon splitting for the spectra of γ - ray pulsars and soft γ - repeaters.

6. Yukawa and Axial Couplings

Up to now we have been concentrating almost exclusively on QED and QCD amplitudes. This reflects the present state-of-the-art, since almost all nontrivial applications of the string-inspired technique have been to these theories. In fact, until recently worldline path integral representations for more general theories were not available in the literature. It will be recalled that, with the exception of the gluon loop case, the worldline path integrals which we have used for QED and QCD have essentially been known since the sixties. It is thus quite surprising that some relatively straightforward extensions of these formulas were apparently never considered until the recent revival of this subject triggered by the work of Bern and Kosower, and Strassler.

Obviously, to be able to treat arbitrary fermion loop processes in the standard model one would need a worldline representation for the coupling of a spin $\frac{1}{2}$ - loop to a more general background including a scalar field ϕ, pseudoscalar field ϕ_5, vector field A, and axialvector field A_5. In (Minkowski space) field theory we would thus be dealing with the following action:

$$S[\phi,\phi_5,A,A_5] = -\int dx\,\bar\psi\Bigl[\not\partial + \phi + i\gamma^5\phi_5 + i\not A + i\gamma^5\not A_5\Bigr]\psi \tag{6.1}$$

Here we absorbed the coupling constants into the background fields. The corresponding Euclidean effective action is given by

$$\Gamma_{\rm E}[\phi,\phi_5,A,A_5] = \ln \mathrm{Det}\Bigl[\not p_E - i\phi + \gamma_{E5}\phi_5 + \not A_E + \gamma_{E5}\not A_{E5}\Bigr] \tag{6.2}$$

In the following we work in Euclidean space-time as usual and drop the subscript E.

In contrast to the QED or QCD effective action, which develops an imaginary part only due to threshold or nonperturbative effects, in the presence of axial vectors or pseudoscalars the effective action can become imaginary already in Euclidean perturbation theory. To be precise, in Euclidean space-time Feynman graphs with an even (odd) number of γ_5 - vertices contribute to $Re(\Gamma_{\rm E})$ $(Im(\Gamma_{\rm E}))$.

A worldline path integral representation for this effective action was constructed in [101,103] for the abelian case, though in a heuristic way. In [102,104] a completely rigorous treatment was given which also includes the antisymmetric tensor coupling and the non-abelian case. A detailed treatment of the general case would be lengthy. We will therefore restrict ourselves to two special cases of particular interest, the scalar-pseudoscalar and the vector – axialvector backgrounds. Moreover, we will take all fields to be abelian, and refer again to [102,104] for the nonabelian generalization.

6.1. Yukawa Couplings from Gauge Theory

For a beginning, let us restrict our attention to the case of only a scalar and a pseudoscalar field. Here it is possible to give a simple and instructive solution of the problem [101], using a dimensional reduction procedure.

As usual in this formalism we try to stay in line with string theory. Now in string theory Yukawa couplings are usually generated in the process of compactifying some of the unphysical dimensions. Our ansatz for this worldline action is therefore to take ϕ_5 and ϕ as the fifth and sixth components of a Yang-Mills field in six dimensions, with the other four components

vanishing, $A = (0,0,0,0,\phi_5,\phi)$. It was already mentioned that the path integral representation eq.(1.9) for the coupling of the spinor loop to a background gauge field remains valid for any even spacetime dimension. In six dimensions and for an A – field of the form above, the worldline Lagrangian becomes

$$\begin{aligned} L &= \frac{1}{4}\dot{x}^2 + \frac{1}{4}\dot{x}_5^2 + \frac{1}{4}\dot{x}_6^2 + \frac{1}{2}\psi\dot{\psi} + \frac{1}{2}\psi_5\dot{\psi}_5 + \frac{1}{2}\psi_6\dot{\psi}_6 + ig\dot{x}_5\phi_5 + ig\dot{x}_6\phi \\ &\quad -2ig\psi^\mu\psi_5\partial_\mu\phi_5 - 2ig\psi^\mu\psi_6\partial_\mu\phi \end{aligned} \tag{6.3}$$

where g denotes the Yang-Mills coupling. We assume ϕ and ϕ_5 to depend only on the four physical dimensions, so that the index μ runs only from 1 to 4. The six-dimensional path integral is then Gaussian in the coordinate fields x_5, x_6. Integrating those out we obtain a new Lagrangian,

$$L = \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} + \frac{1}{2}\psi_5\dot{\psi}_5 + \frac{1}{2}\psi_6\dot{\psi}_6 + g^2\phi^2 + g^2\phi_5^2 - 2ig\psi^\mu\psi_5\partial_\mu\phi_5 - 2ig\psi^\mu\psi_6\partial_\mu\phi \tag{6.4}$$

So far the loop fermion was taken massless (which implies, in particular, that we cannot yet distinguish between the scalar and the pseudoscalar fields). To generate a mass term for the loop fermion we now use the scalar field as a Higgs field, i.e. we give it a non-vanishing vacuum expectation value by shifting

$$\phi \to \phi + \frac{m}{g} \tag{6.5}$$

Moreover, since we do not insist on gauge invariance, we can choose different couplings λ, λ_5 for ϕ, ϕ_5. Our final result for the worldline Lagrangian then becomes

$$\begin{aligned} L_{\rm yuk} &= m^2 + \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi} + \frac{1}{2}\psi_5\dot{\psi}_5 + \frac{1}{2}\psi_6\dot{\psi}_6 + \lambda^2\phi^2 + 2m\lambda\phi + 2i\lambda\psi_6\psi\cdot\partial\phi \\ &\quad +\lambda_5^2\phi_5^2 + 2i\lambda_5\psi_5\psi\cdot\partial\phi_5 \end{aligned} \tag{6.6}$$

This Lagrangian contains two new worldline fields, ψ_5 and ψ_6. The Wick contractions for those are the same as for the other ψ – components,

$$\langle \psi_{5,6}(\tau_1)\,\psi_{5,6}(\tau_2)\rangle = \frac{1}{2}\,G_F(\tau_1,\tau_2) \tag{6.7}$$

Their free path integrals are normalized to unity. Note also the presence of several nonlinear terms in this worldline Lagrangian. For momentum space amplitude calculations those have to be treated in the same way as in the case of ϕ^4 theory above (see section 4.1).

This Lagrangian looks certainly less compelling than its gauge theory analogue. Nevertheless, precisely the same Lagrangian was also obtained by D'Hoker and Gagné in their more rigorous derivation [102,104,182,183]. For now, let us shortly explore its practical usefulness for amplitude calculations.

6.2. N Scalar / N Pseudoscalar Amplitudes

Consider the one-loop one-particle-irreducible amplitude involving either N massless scalars or N massless pseudoscalars, interacting with a fermion loop via Yukawa interactions. Using the above worldline Lagrangian in the usual procedure one finds that a master formula for this amplitude can be obtained from the one for the N - photon amplitude, eq.(4.32), by the following modifications:

1. Write eq.(4.32) in $D = 5$, but with the same path integral determinant factor $(4\pi T)^{-2}$ as in four dimensions.
2. Take all polarization vectors $\tilde{\varepsilon}_i$ in the unphysical dimension, $\tilde{\varepsilon}_i = (0,0,0,0,1)$, and all momenta in the physical dimensions, $\tilde{k}_i = (k_i, 0)$.
3. Delete the constant term in the second derivative of the bosonic worldline Green's function,

$$\ddot{G}_{Bij} = 2\delta_{ij} - \frac{2}{T} \quad \rightarrow \quad 2\delta_{ij} \tag{6.8}$$

Since point 2 implies that all five - dimensional Lorentz products $\tilde{\varepsilon}_i \cdot \tilde{k}_j$ vanish we can write

$$\begin{aligned}\Gamma^{\phi(5)}_{\rm yuk}[k_1,\dots,k_N] &= -2(i\lambda_{(5)})^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_i \\ &\times \exp\Bigl\{\sum_{i,j=1}^N \Bigl[\frac{1}{2}(G_{Bij} + \theta_i\theta_j G_{Fij})k_i\cdot k_j - \frac{1}{2}(G_{Fij} + \theta_i\theta_j 2\delta_{ij})\tilde{\varepsilon}_i\cdot\tilde{\varepsilon}_j\Bigr]\Bigr\} \,|_{\tilde{\varepsilon}_1\dots\tilde{\varepsilon}_N}\end{aligned} \tag{6.9}$$

Here the $\tilde{\varepsilon}_i \cdot \tilde{\varepsilon}_j$ are all equal to unity, but before making use of this the exponential must be expanded, and the $\tilde{\varepsilon}_i$'s anticommuted to the standard ordering $\tilde{\varepsilon}_1 \dots \tilde{\varepsilon}_N$.

In the massive case we have to distinguish between the scalar and pseudoscalar cases. The simpler one is the pseudoscalar case, since according to eq.(6.6) here the only difference between the massless and massive cases is in the usual m^2 - term. Thus all that is needed to generalize the master formula eq.(6.9) to the massive loop case is to supply it with the usual proper-time exponential e^{-m^2T}:

$$\begin{aligned}\Gamma^{\phi_5}_{\rm yuk}[k_1,\dots,k_N] &= -2(i\lambda_5)^N (2\pi)^D \delta(\sum k_i) \int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}} \prod_{i=1}^N \int_0^T d\tau_i \int d\theta_i \\ &\times \exp\Bigl\{\sum_{i,j=1}^N \Bigl[\frac{1}{2}(G_{Bij} + \theta_i\theta_j G_{Fij})k_i\cdot k_j - \frac{1}{2}(G_{Fij} + \theta_i\theta_j 2\delta_{ij})\tilde{\varepsilon}_i\cdot\tilde{\varepsilon}_j\Bigr]\Bigr\} \,|_{\tilde{\varepsilon}_1\dots\tilde{\varepsilon}_N}\end{aligned} \tag{6.10}$$

In the scalar case we have the same mass term, but also the term $2\lambda m\phi$. After the usual formal exponentiation it produces an additional term in the master exponent, so that the massive

master formula for the scalar case becomes somewhat more complicated than the pseudoscalar one:

$$
\begin{aligned}
\Gamma^{\phi}_{\rm yuk}[k_1,\ldots,k_N] &= -2(i\lambda)^N(2\pi)^D\delta(\sum k_i)\int_0^\infty\frac{dT}{T}\,\mathrm{e}^{-m^2T}(4\pi T)^{-\frac{D}{2}}\prod_{i=1}^N\int_0^T d\tau_i\int d\theta_i\\
&\times\exp\Biggl\{\sum_{i,j=1}^N\Bigl[\frac{1}{2}(G_{Bij}+\theta_i\theta_jG_{Fij})k_i\cdot k_j-\frac{1}{2}(G_{Fij}+\theta_i\theta_j2\delta_{ij})\tilde\varepsilon_i\cdot\tilde\varepsilon_j\Bigr]+2im\sum_{i=1}^N\tilde\varepsilon_i\theta_i\Biggr\}\,|_{\tilde\varepsilon_1\ldots\tilde\varepsilon_N}
\end{aligned}
\tag{6.11}
$$

Let us look explicitly at the two-point cases. For the scalar case we get from eq.(6.11) the parameter integral

$$
\begin{aligned}
\Gamma[k_1,k_2] &= -2\lambda^2(2\pi)^D\delta(k_1+k_2)\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\int_0^T d\tau_1\int_0^T d\tau_2\int d\theta_1\int d\theta_2\\
&\times\Bigl[\Bigl(1+\theta_1\theta_2G_{F12}k_1\cdot k_2\Bigr)\Bigl(G_{F12}+\theta_1\theta_22\delta_{12}\Bigr)-4m^2\theta_1\theta_2\Bigr]\mathrm{e}^{G_{B12}k_1\cdot k_2}
\end{aligned}
\tag{6.12}
$$

As usual we rescale to the unit circle, and set $\tau_2=0$. Performing the θ - and T - integrals, and using energy-momentum conservation, we obtain

$$
\Gamma(k)=2(4\pi)^{-\frac{D}{2}}\lambda^2\Biggl\{2\Gamma(1-\frac{D}{2})m^{D-2}-(4m^2+k^2)\Gamma(2-\frac{D}{2})\int_0^1du\Bigl[m^2+u(1-u)k^2\Bigr]^{\frac{D}{2}-2}\Biggr\}
\tag{6.13}
$$

The two - point function for the pseudoscalar case is obtained simply by deleting the term proportional to $4m^2$ (and replacing λ by λ_5).

In both cases the parameter integrals which we have at hand can be identified with the corresponding field theory Feynman parameter integrals, albeit only after suitable integrations by parts. This generalizes, of course, to the N - point case. However, it should be noted that our parameter integrals have a certain advantage insofar as they are already of the scalar type. The master formula involves, apart from the usual factor of $\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}$ which in the T - integration turns into the Feynman denominator, only $\delta(\tau_i-\tau_j)$ and $\mathrm{sign}(\tau_i-\tau_j)$. After restriction to an ordered sector the Feynman numerators are therefore constants. This is not the case in a straightforward Feynman parameter integral calculation of these amplitudes, where one would generally encounter non-trivial numerator polynomials. For example, the above master formula allows one to write the, say, six - point amplitudes immediately in terms of scalar triangle, box, pentagon, and hexagon integrals, without the need to perform a Passarino-Veltman type reduction. Thus it seems that, for the scalar/pseudoscalar amplitudes, one should use the master formula as it stands, without performing partial integrations.

At this point it must be observed that we have been cheating a bit in the pseudoscalar case. From a Feynman diagram analysis it can be easily seen that, in Euclidean space, the amplitude with a massive fermion loop and any number of scalar and pseudoscalar legs is real (imaginary) for an even (odd) number of pseudo-scalars. Since our master formula (6.10) obviously vanishes for N odd, we have seemingly lost the imaginary part of the pseudoscalar amplitude. This was to be expected, since in our heuristic derivation we started from the gauge theory amplitude

in six dimensions, which is real in Euclidean space. The missing imaginary part can also be represented on the worldline, though in a somewhat less natural way [101,102,103,104]. In [184] the resulting path integral representation was applied to the calculation of the radiative decay of the axion into two photons in a constant electromagnetic field, and moreover generalized to the finite temperature case.

6.3. The Spinor Loop in a Vector and Axialvector Background

Another generalization of obvious interest is the inclusion of axialvectors. Here we will not follow the approach taken in [102,104], based on the introduction of auxiliary dimensions, but a more direct construction, which was proposed in [105] and further elaborated in [156,185]. This will also allow us to avoid the separation into the real and the imaginary part of the effective action which was implied in the approach of [102,104].

Thus we would now like to find a path integral representation for (in Euclidean space)

$$\Gamma_{\rm spin}[A,A_5] = \ln \mathrm{Det}[\not p + \not A + \gamma_5 \not A_5 - im] \tag{6.14}$$

The method is a straightforward generalization of the one used in section 3.3 for the pure vector case, and we will indicate only the necessary changes.

First, eq. (3.15) can be generalized to

$$(\not p + \not A + \gamma_5 \not A_5)^2 = -(\partial_\mu + i\mathcal{A}_\mu)^2 + V \tag{6.15}$$

where

$$\begin{aligned} \mathcal{A}_\mu &\equiv A_\mu - \gamma_5 \sigma_{\mu\nu} A_5^\nu && (6.16)\\ V &\equiv -\frac{i}{2}\sigma_{\mu\nu}\Bigl(\partial_\mu A_\nu - \partial_\nu A_\mu\Bigr) + i\gamma_5 A^\mu_{5,\mu} + (D-2)A_5^2 && (6.17) \end{aligned}$$

Here we have used the four - dimensional Dirac algebra, but dimensionally continued with an anticommuting γ_5. Using

$$\mathrm{Det}\Bigl[\not p + \not A + \gamma_5 \not A_5 - im\Bigr] = \mathrm{Det}\Bigl[\not p + \not A + \gamma_5 \not A_5 + im\Bigr] = \mathrm{Det}^{1/2}\Bigl[(\not p + \not A + \gamma_5 \not A_5)^2 + m^2\Bigr] \tag{6.18}$$

one obtains

$$\Gamma_{\rm spin}[A,A_5] = -\frac{1}{2}\,\mathrm{Tr}\int_0^\infty \frac{dT}{T}\,\exp\Bigl\{-T\Bigl[-(\partial_\mu + i\mathcal{A}_\mu)^2 + V + m^2\Bigr]\Bigr\} \tag{6.19}$$

Up to a global factor, this is formally identical with the effective action for a scalar loop in a background containing a (Clifford algebra valued) gauge field $\mathcal{A}$ and a potential V. Note that the exponent is not hermitian, which is the price we have to pay for writing down the whole effective action in one piece. However it is still positive for weak background fields, which is sufficient for our perturbative purposes.

Applying the coherent state formalism in the same way as in section 3.3 we arrive at the following representation, corresponding to (3.27),

$$
\begin{aligned}
\mathrm{Tr}\,\mathrm{e}^{-T\Sigma} &= i\int d^4x \int d^2\eta\,\langle x,-\eta \mid \mathrm{e}^{-T\Sigma} \mid x,\eta\rangle \\
&= i^N \int \prod_{i=1}^{N}\Bigl(d^4x^i d^2\eta^i \langle x^i,\eta^i \mid \mathrm{e}^{-\frac{T}{N}\Sigma} \mid x^{i+1},\eta^{i+1}\rangle\Bigr)
\end{aligned}
\tag{6.20}
$$

where now $\Sigma = -(\partial_\mu + i\mathcal{A}_\mu)^2 + V$. The only essential novelty is the presence of the γ_5 - matrix. To take it into account, it is crucial to observe that, expressed in terms of the $a_r^\pm$, it is identical to the fermion number counter or "G-parity operator" $(-1)^F$ [56,5],

$$
\gamma_5 = (-1)^F = (1-2F_1)(1-2F_2) \tag{6.21}
$$

where

$$
F \equiv F_1 + F_2, \quad \text{with} \quad F_i = a_i^+ a_i^- \tag{6.22}
$$

From the identity

$$
\langle -\eta \mid (-)^F = i\langle 0 \mid \prod_{r=1}^{2}(-\eta_r - a_r^-)(1-2a_r^+a_r^-) = i\langle 0 \mid \prod_{r=1}^{2}(-\eta_r + a_r^-) = \langle \eta \mid \tag{6.23}
$$

it is clear that the presence of $(-1)^F$ can be taken into account by switching the boundary conditions on the Grassmann path integral from antiperiodic to periodic. Thus (3.30) generalizes to

$$
\begin{aligned}
\langle x^i,\eta^i \mid \mathrm{e}^{-\frac{T}{N}\Sigma[p,A,A_5,\gamma_\mu\gamma_\nu,\gamma_5]} \mid x^{i+1},\eta^{i+1}\rangle &= -\frac{i}{(2\pi)^4}\int d^4p^{i,i+1}d^2\bar{\eta}^{i,i+1}\mathrm{e}^{i(x^i-x^{i+1})p^{i,i+1}+(\eta^i-\eta^{i+1})_r\bar{\eta}_r^{i,i+1}} \\
&\quad \times\Bigl\{1-\frac{T}{N}\Sigma\Bigl[p^{i,i+1},A^{i,i+1},A_5^{i,i+1},2\,{}^i\psi_\mu\psi_\nu^{i+1},(-1)^F\Bigr]+O\Bigl(\frac{T^2}{N^2}\Bigr)\Bigr\}
\end{aligned}
\tag{6.24}
$$

In the continuum limit this leads to

$$
\begin{aligned}
\Gamma_{\rm spin}[A,A_5] &= -\frac{1}{2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int \mathcal{D}x\int_A \mathcal{D}\psi\,\mathrm{e}^{-\int_0^T d\tau\,L_{\rm VA}} \\
L_{\rm VA} &= \frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\cdot\dot{\psi} + i\dot{x}^\mu A_\mu - i\psi^\mu F_{\mu\nu}\psi^\nu - 2i\hat{\gamma}_5\dot{x}^\mu\psi_\mu\psi_\nu A_5^\nu + i\hat{\gamma}_5\partial_\mu A_5^\mu + (D-2)A_5^2
\end{aligned}
\tag{6.25}
$$

The operator $(-1)^F$ has turned into an operator $\hat{\gamma}_5$ whose only raison d'être is to determine the boundary conditions of the Grassmann path integral; after expansion of the interaction exponential a given term will have to be evaluated using antiperiodic (periodic) boundary conditions on $\mathcal{D}\psi$, if it contains $\hat{\gamma}_5$ at an even (odd) power. Once the boundary conditions are determined $\hat{\gamma}_5$ can be replaced by unity.

The perturbative evaluation of this double path integral can be done in the usual way. For the coordinate path integral everything proceeds as before. But for the Grassmann path integral one now has to proceed differently depending on the boundary conditions. In the antiperiodic case ("A") there is again nothing new, we can compute it using the by now familiar Green's function G_F. In the periodic case ("P") however one now encounters a fermionic zero mode. As for the coordinate path integral we first must remove this zero mode before executing the path integral. Analogously to eq.(4.4) we can do this by factorizing the Hilbert space of periodic Grassmann functions into the constant functions ψ_0 and their orthogonal complement $\xi(\tau)$,

$$
\begin{aligned}
\int_P \mathcal{D}\psi &= \int d\psi_0 \int \mathcal{D}\xi \\
\psi^\mu(\tau) &= \psi_0^\mu + \xi^\mu(\tau) \\
\int_0^T d\tau\, \xi(\tau) &= 0
\end{aligned}
\tag{6.26}
$$

The zero mode integration then produces the expected ε - tensor via

$$
\int d^4\psi_0 \psi_0^\mu \psi_0^\nu \psi_0^\kappa \psi_0^\lambda = \varepsilon^{\mu\nu\kappa\lambda}
\tag{6.27}
$$

and the ξ - path integral can be performed using the correlator

$$
\langle \xi^\mu(\tau_1)\, \xi^\nu(\tau_2)\rangle \;=\; g^{\mu\nu}\Bigl(\frac{1}{2}\mathrm{sign}(\tau_1-\tau_2) - \frac{\tau_1-\tau_2}{T}\Bigr) = g^{\mu\nu}\frac{1}{2}\dot{G}_{B12}
\tag{6.28}
$$

The free ξ - path integral is, in four dimensions, normalized to unity. Summarizing, in the vector-axialvector case we have the Wick contraction rules

$$
\begin{aligned}
\langle y^\mu(\tau_1)\, y^\nu(\tau_2)\rangle &= -g^{\mu\nu} G_B(\tau_1,\tau_2) \\
\langle \psi^\mu(\tau_1)\, \psi^\nu(\tau_2)\rangle_A &= g^{\mu\nu}\frac{1}{2} G_F(\tau_1,\tau_2) \\
\langle \xi^\mu(\tau_1)\, \xi^\nu(\tau_2)\rangle_P &= g^{\mu\nu}\frac{1}{2} \dot{G}_B(\tau_1,\tau_2)
\end{aligned}
\tag{6.29}
$$

and the free path integral determinants

$$
\begin{aligned}
\int \mathcal{D}y\, \mathrm{e}^{-\int_0^T d\tau\, \frac{1}{4}\dot{y}^2} &= (4\pi T)^{-\frac{D}{2}} \\
\int_A \mathcal{D}\psi\, \mathrm{e}^{-\int_0^T d\tau\, \frac{1}{2}\psi\cdot\dot{\psi}} &= 4 =: N_A \\
\int_P \mathcal{D}\xi\, \mathrm{e}^{-\int_0^T d\tau\, \frac{1}{2}\xi\cdot\dot{\xi}} &= 1 =: N_P
\end{aligned}
\tag{6.30}
$$

6.4. Master Formula for the One-Loop Vector-Axialvector Amplitudes

To extract the scattering amplitude from the effective action, as usual we must specialize the background fields to plane waves, and then keep the part of the effective action which is linear in all polarization vectors. As a preliminary step, it is convenient to linearize the term quadratic in A_5 by introducing an auxiliary path integration, writing

$$\exp\Bigl[-(D-2)\int_0^T d\tau A_5^2\Bigr] \;=\; \int \mathcal{D}z \exp\Bigl[-\int_0^T d\tau\Bigl(\frac{z^2}{4}+i\sqrt{D-2}\hat{\gamma}_5\, z\cdot A_5\Bigr)\Bigr] \qquad (6.31)$$

The Wick contraction rule for this auxiliary field is simply

$$\langle z^\mu(\tau_1)z^\nu(\tau_2)\rangle \;=\; 2g^{\mu\nu}\delta(\tau_1-\tau_2) \qquad (6.32)$$

and its free path integral is normalized to unity. This allows us to define an axial-vector vertex operator as follows,

$$V^{A_5}[k,\varepsilon] \;\equiv\; \hat{\gamma}_5\int_0^T d\tau\Bigl(i\varepsilon\cdot k+2\varepsilon\cdot\psi\dot{x}\cdot\psi+\sqrt{D-2}\,\varepsilon\cdot z\Bigr)\,\mathrm{e}^{ik\cdot x} \qquad (6.33)$$

Before using this vertex operator for Wick-contractions, as usual it is convenient to formally rewrite it as a linearized exponential,

$$\begin{aligned} V^{A_5}[k,\varepsilon] \;&=\; \hat{\gamma}_5\int_0^T d\tau\int d\theta\exp\Bigl[ik\cdot x+i\theta\varepsilon\cdot k+\sqrt{2}\varepsilon\cdot\psi+\sqrt{2}\theta\dot{x}\cdot\psi \\ &\qquad +\sqrt{D-2}\,\theta\varepsilon\cdot z\Bigr]\,|_{\mathrm{lin}(\varepsilon)} \end{aligned} \qquad (6.34)$$

Here θ is a Grassmann variable with $\int d\theta\theta=1$, and ε must now also be formally treated as Grassmann. The vectors are represented by the usual photon vertex operator (4.21)

$$\begin{aligned} V^A[k,\varepsilon] \;&=\; \int_0^T d\tau\Bigl[\varepsilon\cdot\dot{x}+2i\varepsilon\cdot\psi k\cdot\psi\Bigr]\,\mathrm{e}^{ik\cdot x} \\ &=\; \int_0^T d\tau\int d\theta\exp\Bigl[ik\cdot(x+\sqrt{2}\theta\psi)+\varepsilon\cdot(-\theta\dot{x}+\sqrt{2}\psi)\Bigr]\,|_{\mathrm{lin}(\varepsilon)} \end{aligned} \qquad (6.35)$$

Those definitions allow us to represent the one-loop amplitude with M vectors and N axialvectors in the following way,

$$\begin{aligned} \Gamma[\{k_i,\varepsilon_i\},\{k_{5j},\varepsilon_{5j}\}] \;&=\; -\frac{1}{2}N_{A,P}(-i)^{M+N}\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}} \\ &\quad\times\Bigl\langle V^A[k_1,\varepsilon_1]\dots V^A[k_M,\varepsilon_M]V^{A_5}[k_{51},\varepsilon_{51}]\dots V^{A_5}[k_{5N},\varepsilon_{5N}]\Bigr\rangle \end{aligned} \qquad (6.36)$$

where the global sign refers to the ordering $\varepsilon_1\varepsilon_2\dots\varepsilon_M\varepsilon_{51}\varepsilon_{52}\dots\varepsilon_{5N}$ of the polarization vectors. It is then straightforward to perform the bosonic path integrations [20],

[20] In [156] the factor of 2 in front of the term involving $(D-2)\delta(\tau_i-\tau_j)$ had been missing in eqs. (2.14),(2.19), and (2.22).

$$
\begin{aligned}
&\int \mathcal{D}x \int \mathcal{D}z\, V^A[k_1,\varepsilon_1]\dots V^{A_5}[k_{5N},\varepsilon_{5N}]\,\mathrm{e}^{-\int_0^T d\tau\left(\frac{\dot x^2}{4}+\frac{z^2}{4}\right)} = \\
&(4\pi T)^{-\frac{D}{2}} \int_0^T d\tau_1 \cdots \int d\theta_M \int_0^T d\tau_{51} \cdots \int d\theta_{5N} \\
&\times \exp\biggl\{ \frac{1}{2} G_{BIJ} K_I\cdot K_J - i\theta_i \dot G_{BiJ}\varepsilon_i\cdot K_J - i\sqrt{2}\theta_{5i}\dot G_{BiJ}\psi_i\cdot K_J \\
&-\frac{1}{2}\ddot G_{Bij}\theta_i\theta_j\varepsilon_i\cdot\varepsilon_j - \sqrt{2}\ddot G_{Bij}\theta_i\theta_{5j}\varepsilon_i\cdot\psi_j - \ddot G_{Bij}\theta_{5i}\theta_{5j}\psi_i\cdot\psi_j \\
&+\sqrt{2}(\varepsilon_i\cdot\psi_i + \varepsilon_{5j}\cdot\psi_j) + \sqrt{2}i\theta_i k_i\cdot\psi_i + i\theta_{5i}\varepsilon_{5i}\cdot k_{5i} \\
&-2(D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\cdot\varepsilon_{5j}\biggr\}\Big|_{\mathrm{lin}(\{\varepsilon_i\};\{\varepsilon_{5j}\})}
\end{aligned}
\tag{6.37}
$$

Here and in the following all lower case repeated indices run over either the vector or the axial vector indices, depending on whether θ_i, ε_i or $\theta_{5i}, \varepsilon_{5i}$ are involved, while capital repeated indices run over all of them ($\{K_I\}$ denotes the set of all external momenta). We omit the momentum conservation factor (4.13). The remaining ψ - path integral is still Gaussian. For its performance we must now distinguish between even and odd numbers of axialvectors. For the antiperiodic case, N even, there is no zero-mode, and the integration can still be done in closed form. The only complication is the existence of the term $-\ddot G_{Bij}\theta_{5i}\theta_{5j}\psi_i\cdot\psi_j$. It modifies the worldline propagator G_F to

$$
G_{F12}^{(N)} \;\equiv\; 2\langle\tau_1 \mid \Bigl(\partial + 2B^{(N)}\Bigr)^{-1} \mid \tau_2\rangle \tag{6.38}
$$

where $B^{(N)}$ denotes the operator with integral kernel

$$
B^{(N)}(\tau_1,\tau_2) \;=\; \delta(\tau_1-\tau_i)\theta_{5i}\ddot G_{Bij}\theta_{5j}\delta(\tau_j-\tau_2) \tag{6.39}
$$

($B^{(N)}$ acts trivially on the Lorentz indices, which we suppress in the following). Expanding the right hand side of eq.(6.38) in a geometric series and resumming one obtains a matrix representation for $G_{F12}^{(N)}$,

$$
G_F^{(N)} \;=\; \frac{G_F}{\mathbb{1}+\ddot\Theta G_F} = G_F - G_F\ddot\Theta G_F + \dots \tag{6.40}
$$

Here $\ddot\Theta_{ij}$ is the antisymmetric $N\times N$ matrix with entries $\theta_{5i}\ddot G_{Bij}\theta_{5j}$ (no summation). Moreover, the fermionic path integral determinant changes by a factor

$$
\mathrm{Det}^{\frac{1}{2}}\Bigl(\mathbb{1}+2B^{(N)}\partial^{-1}\Bigr) \;=\; \det(\mathbb{1}+\ddot\Theta G_F)^{\frac{D}{2}} \tag{6.41}
$$

as is easily seen using the $lndet = trln$ - formula (note that on the left hand side we have a functional determinant, on the right hand side the determinant of a $N\times N$ matrix). Using these results the fermionic path integral can be eliminated, yielding the following master formula for this amplitude [156],

$$\begin{aligned}
\Gamma_{\rm even}[\{k_i,\varepsilon_i\};\{k_{5j},\varepsilon_{5j}\}] &= -\frac{N_A}{2}(-i)^{M+N}\int_0^\infty \frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\\
\times\int_0^T d\tau_1\int d\theta_1\cdots\int_0^T d\tau_{5N}\int d\theta_{5N}\,&\det(\mathbb{1}+\ddot{\Theta}G_F)^{\frac{D}{2}}\\
\times\exp\biggl\{\frac{1}{2}G_{BIJ}K_I\cdot K_J &- i\theta_i\dot{G}_{BiJ}\varepsilon_i\cdot K_J-\frac{1}{2}\ddot{G}_{Bij}\theta_i\theta_j\varepsilon_i\cdot\varepsilon_j\\
&-\frac{G^{(N)}_{Fij}}{2}\Bigl(\varepsilon_i+i\theta_ik_i+\varepsilon_{5i}-i\theta_{5i}\dot{G}_{BiR}K_R+\ddot{G}_{Bir}\theta_{5i}\theta_r\varepsilon_r\Bigr)\\
&\cdot\Bigl(\varepsilon_j+i\theta_jk_j+\varepsilon_{5j}-i\theta_{5j}\dot{G}_{BjS}K_S+\ddot{G}_{Bjs}\theta_{5j}\theta_s\varepsilon_s\Bigr)+i\theta_{5i}\varepsilon_{5i}\cdot k_{5i}\\
&-2(D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\cdot\varepsilon_{5j}\biggr\}\,|_{\rm lin}\,(\{\varepsilon_i\};\{\varepsilon_{5j}\})
\end{aligned}\tag{6.42}$$

Let us verify the correctness of this formula for the case of the massive 2-point axialvector function in four dimensions. Expanding out the exponential as well as the determinant factor, and performing the two θ - integrals, we obtain the following parameter integral,

$$\begin{aligned}
\Gamma[k_1,\varepsilon_1;k_2,\varepsilon_2] &= 2\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\int_0^T d\tau_1\int_0^T d\tau_2\\
&\times{\rm e}^{G_{B12}k_1\cdot k_2}\Bigl\{2(D-2)\delta(\tau_1-\tau_2)\varepsilon_1\cdot\varepsilon_2-(D-1)\ddot{G}_{B12}G^2_{F12}\varepsilon_1\cdot\varepsilon_2\\
&-G^2_{F12}\dot{G}^2_{B12}(\varepsilon_1\cdot\varepsilon_2k_1\cdot k_2-\varepsilon_1\cdot k_1\varepsilon_2\cdot k_2)-\varepsilon_1\cdot k_1\varepsilon_2\cdot k_2\Bigr\}
\end{aligned}\tag{6.43}$$

As usual we rescale $\tau_{1,2}=Tu_{1,2}$, and use the translation invariance in τ to set $u_2=0$. Setting also $k=k_1=-k_2$ this leads to

$$\begin{aligned}
\Gamma^{\mu\nu}(k) &= 2\int_0^\infty\frac{dT}{T}\,e^{-m^2T}(4\pi T)^{-\frac{D}{2}}\biggl\{2(D-2)Tg^{\mu\nu}-2(D-1)Tg^{\mu\nu}\\
&+\int_0^1 du\,{\rm e}^{-Tu(1-u)k^2}\Bigl[2(D-1)Tg^{\mu\nu}+(1-2u)^2T^2(g^{\mu\nu}k^2-k^\mu k^\nu)+T^2k^\mu k^\nu\Bigr]\biggr\}
\end{aligned}\tag{6.44}$$

In the massless case the first two terms in braces do not contribute in dimensional regularization, since they are of tadpole type. For the remaining terms both integrations are elementary, and the result is, using Γ - function identities, easily identified with the standard result for the massless QED vacuum polarization.

A suitable integration by part verifies the agreement with field theory also for the massive case. Here the tadpole terms do contribute, and the comparison shows that to get the precise D - dependence of the amplitude, appropriate to dimensional regularization using an anticommuting γ_5, it was essential to keep the explicit D - dependence of the A_5^2 - term in the worldline Lagrangian $L_{\rm VA}$.

For an odd number of axial vectors, we need to go back to eq.(6.37) and replace ψ by $\psi_0+\xi$. The $\mathcal{D}\xi$ - path integral is then executed in the same way as before, but with the propagator G_F changed to $\dot{G}_B$. The final result becomes [156]

$$
\begin{aligned}
\Gamma_{\rm odd}[\{k_i,\varepsilon_i\};\{k_{5j},\varepsilon_{5j}\}] \quad &= \quad -\frac{N_P}{2}(-i)^{M+N}\int_0^\infty \frac{dT}{T}\, e^{-m^2T}(4\pi T)^{-\frac{D}{2}} \\
&\times \int_0^T d\tau_1 \int d\theta_1 \cdots \int_0^T d\tau_{5N}\int d\theta_{5N}\,\det(\mathbb{1}+\ddot{\Theta}\dot{G}_B)^{\frac{D}{2}}\int d^4\psi_0 \\
&\times \exp\Biggl\{\frac{1}{2}G_{BIJ}K_I\cdot K_J - i\theta_i\dot{G}_{BiJ}\varepsilon_i\cdot K_J - \frac{1}{2}\ddot{G}_{Bij}\theta_i\theta_j\varepsilon_i\cdot\varepsilon_j \\
&-\frac{\dot{G}^{(N)}_{Bij}}{2}\Bigl(\varepsilon_i + i\theta_i k_i + \varepsilon_{5i} - i\theta_{5i}\dot{G}_{BiR}K_R + \ddot{G}_{Bir}\theta_{5i}\theta_r\varepsilon_r + \sqrt{2}\ddot{G}_{Bir}\theta_{5i}\theta_{5r}\psi_0\Bigr) \\
&\cdot\Bigl(\varepsilon_j + i\theta_j k_j + \varepsilon_{5j} - i\theta_{5j}\dot{G}_{BjS}K_S + \ddot{G}_{Bjs}\theta_{5j}\theta_s\varepsilon_s + \sqrt{2}\ddot{G}_{Bjs}\theta_{5j}\theta_{5s}\psi_0\Bigr) \\
&+ i\theta_{5i}\varepsilon_{5i}\cdot k_{5i} - 2(D-2)\delta(\tau_i-\tau_j)\theta_{5i}\theta_{5j}\varepsilon_{5i}\cdot\varepsilon_{5j} - i\sqrt{2}\theta_{5i}\dot{G}_{BiJ}\psi_0\cdot K_J \\
&-\sqrt{2}\ddot{G}_{Bij}\theta_i\theta_{5j}\varepsilon_i\cdot\psi_0 + \sqrt{2}(\sum\varepsilon_i + \sum\varepsilon_{5j})\cdot\psi_0 + \sqrt{2}i\theta_i k_i\cdot\psi_0\Biggr\}\,|_{\rm lin}\,(\{\varepsilon_i\};\{\varepsilon_{5j}\})
\end{aligned}
\tag{6.45}
$$

Here $\dot{G}^{(N)}_B$ is defined analogously to eq.(6.40),

$$
\dot{G}^{(N)}_B \quad = \quad \frac{\dot{G}_B}{\mathbb{1}+\ddot{\Theta}\dot{G}_B}
\tag{6.46}
$$

The integrand still depends on the zero-mode ψ_0, which is to be integrated according to eq.(6.27).

6.5. The VVA Anomaly

Any new formalism for calculations involving axialvectors must, of course, be confronted with the existence of the chiral anomaly [186,187]. Let us thus verify that our formulas above correctly reproduce the anomaly for the VVA case. Calculation of the $VV\partial\cdot A$ amplitude, either using the master formula eq.(6.45) or a direct Wick-contraction of eq.(6.36), yields the following parameter integral,

$$
\begin{aligned}
k_3^\rho\langle A^\mu[k_1]A^\nu[k_2]A_5^\rho[k_3]\rangle \quad &= \quad 2\varepsilon^{\mu\nu\kappa\lambda}k_1^\kappa k_2^\lambda\int_0^\infty\frac{dT}{T}(4\pi T)^{-2}\prod_{i=1}^3\int_0^T d\tau_i \\
&\times\exp\Bigl[\Bigl(G_{B12}-G_{B13}-G_{B23}\Bigr)k_1\cdot k_2 - G_{B13}k_1^2 - G_{B23}k_2^2\Bigr] \\
&\times\Bigl\{(k_1+k_2)^2 + (\dot{G}_{B12}+\dot{G}_{B23}+\dot{G}_{B31})(\dot{G}_{B13}-\dot{G}_{B23})k_1\cdot k_2 - (\ddot{G}_{B13}+\ddot{G}_{B23})\Bigr\}
\end{aligned}
\tag{6.47}
$$

Here momentum conservation has been used to eliminate k_3. It must be emphasized that this parameter integral represents the complete three-point amplitude, and thus corresponds to the sum of the two different triangle diagrams in field theory, shown in fig. 18.

Removing the second derivatives $\ddot{G}_{B13}$ ($\ddot{G}_{B23}$) by a partial integration in τ_1 (τ_2), the expression in brackets turns into

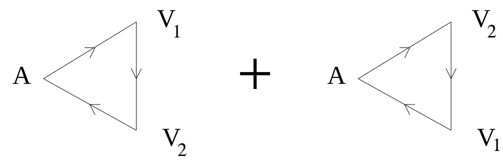


Figure 18: Sum of triangle diagrams in field theory.

$$
\begin{aligned}
&k_1\cdot k_2 \Big\{2-(\dot G_{B12}+\dot G_{B23}+\dot G_{B31})^2+\dot G_{B12}^2-\dot G_{B13}^2-\dot G_{B23}^2\Big\}+k_1^2(1-\dot G_{B13}^2)+k_2^2(1-\dot G_{B23}^2)\\
&=-\frac{4}{T}\Big[\Big(G_{B12}-G_{B13}-G_{B23}\Big)k_1\cdot k_2-G_{B13}k_1^2-G_{B23}k_2^2\Big]
\end{aligned}
\tag{6.48}
$$

In the last step we used the identities (F.25),(F.26). This is precisely the same expression which appears also in the exponential factor in (6.47). After performing the trivial T - integral we find therefore a complete cancellation between the Feynman numerator and denominator polynomials, and obtain without further integration the desired result,

$$
k_3^\rho \langle A^\mu A^\nu A_5^\rho\rangle \quad = \quad \frac{8}{(4\pi)^2}\,\varepsilon^{\mu\nu\kappa\lambda}k_1^\kappa k_2^\lambda
\tag{6.49}
$$

This is the usual expression for the divergence of the axialvector current [186,187]. Note that in the present formalism this divergence is unambiguously fixed to be at the axialvector current. This can, in fact, be already seen at the path integral level, since the vectors are represented by the photon vertex operator (6.35) which, as is familiar from string theory, turns into a total derivative when contracted with its own momentum. This will lead to the vanishing of the whole amplitude, independently of the possible divergence of the global T - integration. Nothing analogous holds true for the axialvector vertex operator.

The behaviour of the present formalism with respect to the chiral symmetry in the dimensional cointinuation is thus somewhat unusual. It should be remembered that, in field theory, one has essentially a choice between two evils. If one preserves the anticommutation relation between γ_5 and the other Dirac matrices [188] then the chiral symmetry is preserved for parity-even fermion loops, but Dirac traces with an odd number of γ_5's are not unambiguously defined in general, requiring additional prescriptions. The main alternative is to use the 't Hooft-Veltman-Breitenlohner-Maison prescription [189,190]. In this case there are no ambiguities, but the chiral symmetry is explicitly broken, so that in chiral gauge theories finite renormalizations generally become necessary to avoid violations of the gauge Ward identities [191].

Since our path integral representation was derived using an anticommuting γ_5, we have not broken the chiral symmetry. In particular, in the massless case the amplitude with an even

number of axialvectors should coincide with the corresponding vector amplitude, and we have explicitly verified this fact for the two-point case. (Even though the structure of the resulting Feynman numerators is quite different from the equivalent ones derived from the ordinary Bern-Kosower master formula.) Nevertheless, we did not encounter any ambiguities even in the parity-odd case, not even in the anomaly calculation. This property of the formalism may be useful for applications to chiral gauge theories.

6.6. Inclusion of Constant Background Fields

As in the pure vector case, it is easy to take into account an additional (vector) background field with constant field strength tensor $F_{\mu\nu}$ [192]. The presence of the background field modifies the Wick contraction rules (6.29) to

$$
\begin{aligned}
\langle y^\mu(\tau_1) y^\nu(\tau_2)\rangle &= -\mathcal{G}_B^{\mu\nu}(\tau_1,\tau_2)\\
\langle \psi^\mu(\tau_1) \psi^\nu(\tau_2)\rangle &= \frac{1}{2}\mathcal{G}_F^{\mu\nu}(\tau_1,\tau_2)\\
\langle \xi^\mu(\tau_1) \xi^\nu(\tau_2)\rangle &= \frac{1}{2}\dot{\mathcal{G}}_B^{\mu\nu}(\tau_1,\tau_2)
\end{aligned}
\tag{6.50}
$$

(compare chapter 5). The Gaussian path integral determinants (6.30) become

$$
\begin{aligned}
\int \mathcal{D}y\, \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{4}\dot{y}^2+\frac{1}{2}ie\,y^\mu F_{\mu\nu}\dot{y}^\nu\right)} &= (4\pi T)^{-\frac{D}{2}}\det{}^{-\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]\\
\int_A \mathcal{D}\psi\, \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{2}\psi\cdot\dot{\psi}-ie\,\psi^\mu F_{\mu\nu}\psi^\nu\right)} &= 4\det{}^{\frac{1}{2}}\left[\cos\mathcal{Z}\right]\\
\int_P \mathcal{D}\xi\, \mathrm{e}^{-\int_0^T d\tau\left(\frac{1}{2}\xi\cdot\dot{\xi}-ie\,\xi^\mu F_{\mu\nu}\xi^\nu\right)} &= \det{}^{\frac{1}{2}}\left[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\right]
\end{aligned}
\tag{6.51}
$$

Note that for the case of periodic Grassmann boundary conditions the field dependence of the determinant factors cancels out between the coordinate and Grassmann path integrals. This cancellation is a consequence of the worldline supersymmetry (1.10) [20,21,64,156]. It does not occur in the antiperiodic case since here the supersymmetry is broken by the boundary conditions (see appendix C of [156] for more on this point).

In the periodic case the external field must, moreover, be also taken into account in the zero-mode integration, as shown in the following example.

6.7. Example: Vector - Axialvector Amplitude in a Constant Field

As an explicit example, we calculate the vector – axialvector two-point function in a constant field. This amplitude is relevant, for example, for photon – neutrino processes at low photon energies (see, e.g., [193,194,171]). According to the above we can represent this amplitude as follows,

$$
\begin{aligned}
\langle A_\mu(k_1)A_{5\nu}(k_2)\rangle \;=\; & \frac{1}{2}\int_0^\infty \frac{dT}{T}\,e^{-m^2T}\int \mathcal{D}x\int_P \mathcal{D}\psi \\
& \times \exp\Big\{-\int_0^T d\tau\Big[\frac{1}{4}\dot{x}^2+\frac{1}{2}\psi\cdot\dot{\psi}+\frac{i}{2}e\,x\cdot F\cdot\dot{x}-ie\,\psi\cdot F\cdot\psi\Big]\Big\} \\
& \times \int_0^T d\tau_1\Big(\dot{x}_\mu(\tau_1)+2i\psi_\mu(\tau_1)k_1\cdot\psi(\tau_1)\Big)\mathrm{e}^{ik_1\cdot x_1} \\
& \times \int_0^T d\tau_2\Big(ik_{2\nu}+2\psi_\nu(\tau_2)\dot{x}(\tau_2)\cdot\psi(\tau_2)\Big)\mathrm{e}^{ik_2\cdot x_2}
\end{aligned}
\tag{6.52}
$$

This amplitude is finite, so that we can set $D = 4$ in its evaluation. As a first step, the zero-modes of both path integrals are separated out according to eqs.(4.4),(6.26), and the Grassmann zero mode integrated out using eq.(6.27). All terms which do not contain all four zero mode components precisely once give zero. To explicitly perform this integration we note that by eq.(6.26) we can rewrite, in the exponent of eq.(6.52),

$$
\int_0^T d\tau\,\psi(\tau)\cdot F\cdot\psi(\tau) = T\psi_0\cdot F\cdot\psi_0+\int_0^T d\tau\,\xi(\tau)\cdot F\cdot\xi(\tau)
\tag{6.53}
$$

Thus for the case at hand the Grassmann zero mode integral can appear in the following three forms,

$$
\begin{aligned}
\int d^4\psi_0\,\mathrm{e}^{ieT\psi_0\cdot F\cdot\psi_0} &= -\frac{(eT)^2}{2}\varepsilon_{\mu\nu\kappa\lambda}F_{\mu\nu}F_{\kappa\lambda} = -(eT)^2F\cdot\tilde{F} \\
\int d^4\psi_0\,\mathrm{e}^{ieT\psi_0\cdot F\cdot\psi_0}\psi_{0\mu}\psi_{0\nu} &= ieT\varepsilon_{\mu\nu\kappa\lambda}F_{\kappa\lambda} = 2ieT\tilde{F}_{\mu\nu} \\
\int d^4\psi_0\,\mathrm{e}^{ieT\psi_0\cdot F\cdot\psi_0}\psi_{0\mu}\psi_{0\nu}\psi_{0\kappa}\psi_{0\lambda} &= \varepsilon_{\mu\nu\kappa\lambda}
\end{aligned}
\tag{6.54}
$$

In the next step, both path integrations are performed using the field-dependent Wick contraction rules eqs. (6.50). This results in the following parameter integral representation for the vector – axialvector vacuum polarization tensor [21]

$$
\begin{aligned}
\Pi_5^{\mu\nu}(k) &= \frac{ee_5}{8\pi^2}\int_0^\infty \frac{dT}{T^3}\,\mathrm{e}^{-m^2T}\int_0^T d\tau_1 d\tau_2\;J_5^{\mu\nu}(\tau_1,\tau_2)\,\mathrm{e}^{-k\cdot\bar{\mathcal{G}}_{12}\cdot k} \\
J_5^{\mu\nu}(\tau_1,\tau_2) &= \Big[\ddot{\mathcal{G}}_{12}^{\mu\alpha}-(\dot{\mathcal{G}}_{21}^{\alpha\beta}-\dot{\mathcal{G}}_{22}^{\alpha\beta})(\dot{\mathcal{G}}_{11}^{\mu\rho}-\dot{\mathcal{G}}_{12}^{\mu\rho})k_\beta k_\rho\Big]\Big(i\tilde{\mathcal{Z}}_{\nu\alpha}-\frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\dot{\mathcal{G}}_{22}^{\nu\alpha}\Big) \\
&+\frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}(\dot{\mathcal{G}}_{11}^{\mu\rho}-\dot{\mathcal{G}}_{12}^{\mu\rho})k_\rho k_\nu+k_\nu k_\rho\Big(i\tilde{\mathcal{Z}}_{\mu\rho}-\frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\dot{\mathcal{G}}_{11}^{\mu\rho}\Big)+k_\rho k_\sigma(\dot{\mathcal{G}}_{21}^{\alpha\rho}-\dot{\mathcal{G}}_{22}^{\alpha\rho}) \\
&\times\Big[\varepsilon_{\mu\sigma\nu\alpha}+i(\dot{\mathcal{G}}_{22}^{\nu\alpha}\tilde{\mathcal{Z}}_{\mu\sigma}-\dot{\mathcal{G}}_{12}^{\sigma\alpha}\tilde{\mathcal{Z}}_{\mu\nu}+\dot{\mathcal{G}}_{12}^{\sigma\nu}\tilde{\mathcal{Z}}_{\mu\alpha}+\dot{\mathcal{G}}_{12}^{\mu\alpha}\tilde{\mathcal{Z}}_{\sigma\nu}-\dot{\mathcal{G}}_{12}^{\mu\nu}\tilde{\mathcal{Z}}_{\sigma\alpha}+\dot{\mathcal{G}}_{11}^{\mu\sigma}\tilde{\mathcal{Z}}_{\nu\alpha}) \\
&-\frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}(\dot{\mathcal{G}}_{11}^{\mu\sigma}\dot{\mathcal{G}}_{22}^{\nu\alpha}-\dot{\mathcal{G}}_{12}^{\mu\nu}\dot{\mathcal{G}}_{12}^{\sigma\alpha}+\dot{\mathcal{G}}_{12}^{\mu\alpha}\dot{\mathcal{G}}_{12}^{\sigma\nu})\Big]
\end{aligned}
\tag{6.55}
$$

[21] Since $G_F, \mathcal{G}_F$ do not occur for the periodic case we delete the subscript "B" in the remainder of this section.

where $k = k_1 = -k_2, \tilde{\mathcal{Z}} \equiv eT\tilde{F}$. As usual it is useful to perform a partial integration on the one term involving $\ddot{\mathcal{G}}_{12}$, leading to the replacement

$$\ddot{\mathcal{G}}_{12}^{\mu\alpha} \to \dot{\mathcal{G}}_{12}^{\mu\alpha} k \cdot \dot{\mathcal{G}}_{12} \cdot k \tag{6.56}$$

By this partial integration, and the removal of some terms which cancel against each other, $J_5^{\mu\nu}(\tau_1, \tau_2)$ gets replaced by

$$\begin{aligned}
&k^\rho k^\sigma \Big[\dot{\mathcal{G}}_{12}^{\mu\alpha}\dot{\mathcal{G}}_{12}^{\rho\sigma} + (\dot{\mathcal{G}}_{21}^{\alpha\sigma} - \dot{\mathcal{G}}_{22}^{\alpha\sigma})\dot{\mathcal{G}}_{12}^{\mu\rho}\Big]\Big(i\tilde{\mathcal{Z}}^{\nu\alpha} - \frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\dot{\mathcal{G}}_{22}^{\nu\alpha}\Big) - \frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\dot{\mathcal{G}}_{12}^{\mu\rho}k^\rho k^\nu \\
&+ik^\nu k^\rho \tilde{\mathcal{Z}}^{\mu\rho} + k^\rho k^\sigma (\dot{\mathcal{G}}_{21}^{\alpha\rho} - \dot{\mathcal{G}}_{22}^{\alpha\rho})\Big[\varepsilon^{\mu\sigma\nu\alpha} + \frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}(\dot{\mathcal{G}}_{12}^{\mu\nu}\dot{\mathcal{G}}_{12}^{\sigma\alpha} - \dot{\mathcal{G}}_{12}^{\mu\alpha}\dot{\mathcal{G}}_{12}^{\sigma\nu}) \\
&+i(\dot{\mathcal{G}}_{22}^{\nu\alpha}\tilde{\mathcal{Z}}^{\mu\sigma} - \dot{\mathcal{G}}_{12}^{\sigma\alpha}\tilde{\mathcal{Z}}^{\mu\nu} + \dot{\mathcal{G}}_{12}^{\sigma\nu}\tilde{\mathcal{Z}}^{\mu\alpha} + \dot{\mathcal{G}}_{12}^{\mu\alpha}\tilde{\mathcal{Z}}^{\sigma\nu} - \dot{\mathcal{G}}_{12}^{\mu\nu}\tilde{\mathcal{Z}}^{\sigma\alpha})\Big]
\end{aligned} \tag{6.57}$$

As in the vector – vector case, we decompose $\mathcal{G}_{ij}$ as

$$\mathcal{G}_{ij} \;=\; \mathcal{S}_{ij} + \mathcal{A}_{ij} \tag{6.58}$$

where $\mathcal{S}$ ($\mathcal{A}$) are its parts even (odd) in F. We can then delete all terms odd in $\tau_1 - \tau_2$ since they vanish upon integration. After using the identity $F\tilde{F} = -g\mathbb{1}$ and some combining of terms, J_5 finally turns into the following, nicely symmetric expression I_5,

$$\begin{aligned}
I_5^{\mu\nu}(\tau_1,\tau_2) \;=\;& i\Big\{\tilde{\mathcal{Z}}^{\mu\nu}k\mathcal{U}_{12}k + \Big[(\tilde{\mathcal{Z}}k)^\mu(\mathcal{U}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Big] \\
&-(\tilde{\mathcal{Z}}\dot{\mathcal{S}}_{12})^{\mu\nu}k\dot{\mathcal{S}}_{12}k - \Big[(\tilde{\mathcal{Z}}\dot{\mathcal{S}}_{12}k)^\mu(\dot{\mathcal{S}}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Big]\Big\} \\
&+\frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\Big\{-\dot{\mathcal{A}}_{12}^{\mu\nu}k\mathcal{U}_{12}k - \Big[(\dot{\mathcal{A}}_{12}k)^\mu(\mathcal{U}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Big] \\
&+(\dot{\mathcal{A}}_{22}\dot{\mathcal{S}}_{12})^{\mu\nu}k\dot{\mathcal{S}}_{12}k + \Big[(\dot{\mathcal{A}}_{22}\dot{\mathcal{S}}_{12}k)^\mu(\dot{\mathcal{S}}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Big]\Big\}
\end{aligned} \tag{6.59}$$

Here in addition to $\mathcal{A}$ and $\mathcal{S}$ we have introduced the combination $\mathcal{U}$,

$$\mathcal{U}_{12} \;=\; \dot{\mathcal{S}}_{12}^2 - (\dot{\mathcal{A}}_{12} - \dot{\mathcal{A}}_{22})(\dot{\mathcal{A}}_{12} + \frac{i}{\mathcal{Z}}) = \frac{1-\cos(\mathcal{Z}\dot{G}_{12})\cos(\mathcal{Z})}{\sin^2(\mathcal{Z})} \tag{6.60}$$

Defining also

$$\hat{\mathcal{A}} \equiv \dot{\mathcal{A}} + \frac{i}{\mathcal{Z}} \tag{6.61}$$

this expression can be further compressed to

$$
\begin{aligned}
I_5^{\mu\nu}(\tau_1,\tau_2) &= \frac{\mathcal{Z}\cdot\tilde{\mathcal{Z}}}{4}\Bigl\{-\hat{\mathcal{A}}_{12}^{\mu\nu}k\mathcal{U}_{12}k - \Bigl[(\hat{\mathcal{A}}_{12}k)^\mu(\mathcal{U}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Bigr] \\
&\quad +(\hat{\mathcal{A}}_{22}\dot{\mathcal{S}}_{12})^{\mu\nu}k\dot{\mathcal{S}}_{12}k + \Bigl[(\hat{\mathcal{A}}_{22}\dot{\mathcal{S}}_{12}k)^\mu(\dot{\mathcal{S}}_{12}k)^\nu + (\mu\leftrightarrow\nu)\Bigr]\Bigr\}
\end{aligned}
\tag{6.62}
$$

We can now use the matrix decompositions of $\mathcal{S},\dot{\mathcal{S}},\dot{\mathcal{A}}$, given in eq.(5.45), to write the integrand in explicit form. In this we have a choice between the matrix bases $\{\hat{\mathcal{Z}}_\pm,\hat{\mathcal{Z}}_\pm^2\}$ or $\{\mathbb{1},F,\tilde{F},F^2\}$. We will use the former one here since it leads to a somewhat more compact expression. After the usual rescaling to the unit circle, a transformation of variables $v=\dot{G}_{12}$, and continuation to Minkowski space, we obtain our final result for the vector – axialvector amplitude in a constant field [192] ,

$$
\begin{aligned}
\Pi_5^{\mu\nu}(k) &= \frac{e^3e_5}{8\pi^2}\mathcal{G}\int_0^\infty ds\, s\,\mathrm{e}^{-ism^2}\int_{-1}^1\frac{dv}{2}\exp\Bigl[-i\frac{s}{2}\sum_{\alpha=+,-}\frac{\hat{A}_{B12}^\alpha-\hat{A}_{B11}^\alpha}{z_\alpha}\,k\cdot\hat{\mathcal{Z}}_\alpha^2\cdot k\Bigr] \\
&\quad\times\sum_{\alpha,\beta=+,-}\Bigl[\hat{A}_{12}^\alpha\Bigl((\hat{A}_{12}^\beta-\hat{A}_{22}^\beta)\hat{A}_{12}^\beta-(S_{12}^\beta)^2\Bigr)+\hat{A}_{22}^\alpha S_{12}^\alpha S_{12}^\beta\Bigr] \\
&\quad\quad\times\Bigl[\hat{\mathcal{Z}}_\alpha^{\mu\nu}k\hat{\mathcal{Z}}_\beta^2k+(\hat{\mathcal{Z}}_\alpha k)^\mu(\hat{\mathcal{Z}}_\beta^2k)^\nu+(\hat{\mathcal{Z}}_\alpha k)^\nu(\hat{\mathcal{Z}}_\beta^2k)^\mu\Bigr]
\end{aligned}
\tag{6.63}
$$

where

$$
\begin{aligned}
S_{12}^\pm &= \frac{\sinh(z_\pm v)}{\sinh(z_\pm)} \\
\hat{A}_{12}^\pm &= \frac{\cosh(z_\pm v)}{\sinh(z_\pm)} \\
\hat{A}_{ii}^\pm &= \coth(z_\pm)
\end{aligned}
\tag{6.64}
$$

and $z_\pm,\hat{\mathcal{Z}}_\pm,a,b$ are the same as in (5.66),(5.67). As in the vector – vector case, this expression becomes somewhat more transparent if one specializes to the Lorentz system where $\mathbf{E}$ and $\mathbf{B}$ are both pointing along the positive z - axis, $\mathbf{E}=(0,0,E),\mathbf{B}=(0,0,B)$. Here one obtains

$$
\Pi_5^{\mu\nu}(k) = i\frac{e^2e_5}{8\pi^2}\int_0^\infty ds\int_{-1}^1\frac{dv}{2}\,\mathrm{e}^{-is\Phi_0}\sum_{\alpha,\beta=\perp,\|}c^{\alpha\beta}\Bigl[\tilde{F}_\alpha^{\mu\nu}k_\beta^2+(\tilde{F}_\alpha k)^\mu k_\beta^\nu+(\tilde{F}_\alpha k)^\nu k_\beta^\mu\Bigr]
\tag{6.65}
$$

where $z=eBs, z'=eEs,\ k_\perp=(0,k^1,k^2,0),\ k_\|=(k^0,0,0,k^3)$,

$$(\tilde{F}_{\|})^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & B \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -B & 0 & 0 & 0 \end{pmatrix}, \qquad (\tilde{F}_{\perp})^{\mu\nu} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.66}$$

$$\Phi_0 = m^2 + \frac{k_\perp^2}{2}\frac{\cos(zv)-\cos(z)}{z\sin(z)} - \frac{k_\|^2}{2}\frac{\cosh(z'v)-\cosh(z')}{z'\sinh(z')} \tag{6.67}$$

$$\begin{aligned} c^{\perp\perp} &= z\frac{\cos(zv)-\cos(z)}{\sin^3(z)} \\ c^{\perp\|} &= \frac{z\cos(zv)}{\sin(z)}\frac{\cosh(z')\cosh(z'v)-1}{\sinh^2(z')} - \frac{z\cos(z)\sin(zv)}{\sin^2(z)}\frac{\sinh(z'v)}{\sinh(z')} \\ c^{\|\perp} &= -\frac{z'\cosh(z'v)}{\sinh(z')}\frac{\cos(zv)\cos(z)-1}{\sin^2(z)} - \frac{z'\cosh(z')\sinh(z'v)}{\sinh^2(z')}\frac{\sin(zv)}{\sin(z)} \\ c^{\|\|} &= -z'\frac{\cosh(z'v)-\cosh(z')}{\sinh^3(z')} \end{aligned} \tag{6.68}$$

This result can still be slightly simplified using the relations

$$\tilde{F}_\alpha^{\mu\nu}k_\alpha^2 = (\tilde{F}_\alpha k)^\mu k_\alpha^\nu - (\tilde{F}_\alpha k)^\nu k_\alpha^\mu \tag{6.69}$$

($\alpha =\perp, \|$). It agrees, even at the integrand level, with the recent field theory result of [195]. We remark that via the axial Ward identity

$$k_2^\nu \langle A_\mu(k_1)A_{5\nu}(k_2)\rangle \quad = \quad -2im\langle A_\mu(k_1)\phi_5(k_2)\rangle \tag{6.70}$$

from $\Pi_5^{\mu\nu}$ one can also immediately obtain the vector – pseudoscalar amplitude in a constant field. This amplitude leads, for example, to a field – induced effective axion – photon interaction [196]. More generally, from the Ward identity it is clear that the Lorentz contraction $V^{A_5}[k,k]$ of the axialvector vertex operator (6.33) can effectively serve as a pseudoscalar vertex operator.

7. Effective Actions and their Inverse Mass Expansions

In the previous chapters we have derived worldline path integrals representing effective actions, but applied them mainly to the calculation of scattering amplitudes. In this chapter, we calculate the effective action directly in x – space, in a higher derivative expansion. The method used is a pure x – space version of the one used by Strassler in [150], made manifestly gauge invariant by the use of Fock-Schwinger gauge.

7.1. The Inverse Mass Expansion for Non-Abelian Gauge Theory

The higher derivative expansion is a standard tool for the approximative calculation of one-loop effective actions, and considerable work has gone into the determination of its coefficients for various theories (see [197,198] and refs. therein).

This expansion exists in several versions, which differ by the grouping of terms. The one which we will consider here is the "inverse mass expansion", which is just the expansion in powers of the proper-time parameter T. This groups together terms of equal mass dimension. Up to partial integrations in space-time, it coincides with the (diagonal part of the) "heat kernel expansion" for the second order differential operator in question. In particular, every coefficient in this expansion is separately gauge invariant.

Alternatively, one may calculate the same series up to a fixed number of derivatives, but with an arbitrary number of fields or potentials [199,200,201,202,142]. In QED this corresponds, to zeroeth order, to the approximation of the effective Lagrangian by the Euler-Heisenberg Lagrangian (see chapter 5). See [97,98] for a calculation of the first gradient correction to the Euler-Heisenberg Lagrangian.

Yet another option is to keep the number of external fields fixed, and sum up the derivatives to all orders. This leads to the notion of Barvinsky-Vilkovisky form factors [203,204,205]. For the calculation of some such form factors in the string-inspired formalism see [96].

Individual terms in this expansion are also relevant for the determination of counterterms in the corresponding field theories, defined at a spacetime dimension D which is related to the mass dimension of the term considered [206].

We consider a background consisting of a scalar field and/or a gauge field, both possibly non-abelian. In this background, the scalar loop path integral (1.8) generalizes to

$$\Gamma_{\rm scal}[A,V] = \int_0^\infty \frac{dT}{T}\, e^{-m^2T}\,{\rm tr}\int\mathcal{D}x\,\exp\Big[-\int_0^T d\tau\Big(\frac{1}{4}\dot{x}^2 + ig\dot{x}\cdot A + V\Big)\Big] \tag{7.1}$$

Here we have rewritten $V(x) \equiv U''(\phi(x))$, where $U(\phi)$ is the field theory interaction potential of chapter 3. The path integral is path-ordered except if both A and V are abelian. in most applications of the heat kernel expansion the loop spin can be taken into account by an appropriate choice of the scalar part V (see, e.g., [207,208]). In this context we will therefore restrict ourselves to a treatment of the scalar loop case, although the evaluation technique outlined below extends to the spinor and gluon path integrals in an obvious way; see [96,97,98] for the case of the fermion loop in QED. For QED this approach to the calculation of the effective action has also been generalized to the finite temperature case [162].

As always we separate out the ordinary integral over the loop center of mass x_0, which reduces

the effective action to the effective Lagrangian,

$$\Gamma[A,V] = \int dx_0\,\mathcal{L}[A,V](x_0)$$

To obtain the higher derivative expansion, we Taylor–expand both A and V at x_0,

$$\begin{aligned} V(x) &= e^{y\cdot\partial}V(x_0) \\ \dot{x}^\mu A_\mu(x) &= \dot{y}^\mu e^{y\cdot\partial}A_\mu(x_0) \end{aligned} \tag{7.2}$$

The path-ordered interaction exponential is then expanded to yield

$$\begin{aligned} \Gamma_{\rm scal}[A,V] &= {\rm tr}\int_0^\infty \frac{dT}{T}e^{-m^2T}\int d^Dx_0\sum_{n=0}^\infty\frac{(-1)^n}{n}T\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\dots\int_0^{\tau_{n-2}}d\tau_{n-1} \\ &\times \int\mathcal{D}y\Big[ig\,\dot{y}^{\mu_1}(\tau_1)e^{y(\tau_1)\partial_{(1)}}A^{(1)}_{\mu_1}(x_0)+e^{y(\tau_1)\partial_{(1)}}V^{(1)}(x_0)\Big]\dots \\ &\times \Big[ig\,\dot{y}^{\mu_n}(\tau_n)e^{y(\tau_n)\partial_{(n)}}A^{(n)}_{\mu_n}(x_0)+e^{y(\tau_n)\partial_{(n)}}V^{(n)}(x_0)\Big]\exp\Big[-\int_0^T d\tau\frac{\dot{y}^2}{4}\Big] \end{aligned} \tag{7.3}$$

Here we have labelled the background fields, and fixed $\tau_n = 0$. This is also the origin of the factor of $\frac{1}{n}$. We then use the Wick contraction rules for evaluating the individual terms in this expansion, e. g.,

$$\begin{aligned} \langle e^{y(\tau_1)\partial_{(1)}}e^{y(\tau_2)\partial_{(2)}}\rangle &= e^{-G_B(\tau_1,\tau_2)\partial_{(1)}\cdot\partial_{(2)}} \\ \langle \dot{y}^\mu(\tau_1)e^{y(\tau_1)\partial_{(1)}}e^{y(\tau_2)\partial_{(2)}}\rangle &= -\dot{G}_B(\tau_1,\tau_2)\partial^\mu_{(2)}e^{-G_B(\tau_1,\tau_2)\partial_{(1)}\cdot\partial_{(2)}} \end{aligned} \tag{7.4}$$

As in our earlier constant background field calculations, we can enforce manifest gauge invariance by choosing Fock-Schwinger gauge centred at x_0. The gauge condition is

$$y^\mu A_\mu(x_0+y(\tau)) \equiv 0 \tag{7.5}$$

In this gauge,

$$A_\mu(x_0+y) = y^\rho\int_0^1 d\eta\eta F_{\rho\mu}(x_0+\eta y) \tag{7.6}$$

and $F_{\rho\mu}$ and V can be *covariantly* Taylor-expanded as (see, e.g., [209])

$$\begin{aligned} F_{\rho\mu}(x_0+\eta y) &= e^{\eta y\cdot D}F_{\rho\mu}(x_0) \\ V(x_0+y) &= e^{y\cdot D}V(x_0) \end{aligned} \tag{7.7}$$

This leads also to a covariant Taylor expansion for A:

$$A_\mu(x_0+y) = \int_0^1 d\eta\eta\, y^\rho e^{\eta y\cdot D} F_{\rho\mu}(x_0) = \frac{1}{2}y^\rho F_{\rho\mu} + \frac{1}{3}y^\nu y^\rho D_\nu F_{\rho\mu} + ... \tag{7.8}$$

Using these formulas, we obtain the following manifestly covariant version of eq.(7.3):

$$\begin{aligned}\Gamma_{\rm scal}[F,V] \;\; &= \;\; {\rm tr}\int_0^\infty \frac{dT}{T} e^{-m^2T} \int d^Dx_0 \sum_{n=0}^\infty \frac{(-1)^n}{n} T \int_0^T d\tau_1 \int_0^{\tau_1} d\tau_2 \dots \int_0^{\tau_{n-2}} d\tau_{n-1} \\ &\times \int \mathcal{D}y \exp\Bigl[-\int_0^T d\tau \frac{\dot y^2}{4}\Bigr] \prod_{j=1}^n \Bigl[\, {\rm e}^{y(\tau_j)D_{(j)}} V^{(j)}(x_0) + ig\dot y^{\mu_j}(\tau_j) y^{\rho_j}(\tau_j) \int_0^1 d\eta_j \eta_j {\rm e}^{\eta_j y(\tau_j) D_{(j)}} F^{(j)}_{\rho_j\mu_j}(x_0)\Bigr]\end{aligned} \tag{7.9}$$

From this master formula, the inverse mass expansion to some fixed order N,

$$\Gamma_{\rm scal}[F,V] = \int_0^\infty \frac{dT}{T} \frac{{\rm e}^{-m^2T}}{(4\pi T)^{D/2}}\, {\rm tr} \int dx_0 \sum_{n=1}^N \frac{(-T)^n}{n!}\, O_n[F,V]\,, \tag{7.10}$$

is obtained in three steps:

Wick contractions: Truncate the master formula to $n = N$, and the covariant Taylor expansion eq.(7.8) accordingly. Perform the Wick contractions. Alternatively, one may also first Wick contract the complete expression for $n = N$, and truncate the Taylor expansion afterwards. (This procedure is preferable in the pure scalar field case.)

Integrations: Perform the τ − integrations. The integrand is a polynomial in the worldline Green's function G_B, $\dot G_B$, and $\ddot G_B$. As usual, the τ − integrals can be rescaled to the unit circle, $\tau_i = Tu_i$. The δ-function in $\ddot G_B(u_i, u_j)$ only contributes if u_i and u_j are neighbouring points on the loop. (Note that this includes the case $\ddot G_B(1, u_n)$.) In the non-Abelian case the coefficient 2 in front of the δ-function has to be omitted, since only half of the δ-function contributes to the ordered sector under consideration (in the scattering amplitude context this rule was already stated in section 4.4).

Reduction to a minimal basis: The result of this procedure is the effective Lagrangian at the required order, albeit in redundant form. To be maximally useful for numerical applications, it still needs to be reduced to a minimal set of invariants, using all available symmetries. Those are

1. Cyclic invariance under the trace.
2. Bianchi identities.
3. For real representations of the gauge group one has an additional symmetry under transposition (up to a sign for every factor of $F_{\mu\nu}$).

Usually those symmetry operations would have to be combined with judiciously chosen partial integrations performed on the effective Lagrangian. It is a remarkable property of the present

calculational scheme that the reduction of our result for the effective action to a minimal basis of invariants can be achieved without any such partial integrations. In particular, for the pure scalar case the reduction process amounts to nothing more than the identification of cyclically equivalent terms (in fact, for this special case the whole procedure can be condensed into a purely combinatorial formula [210]). In the general case, the reduction to a minimal basis of invariants is much more involved. The method adopted here follows a proposal by Müller [211,212], which is also explained in [107].

Let us give the explicit result of this procedure up to order $O(T^4)$ (absorbing the coupling constant g into the fields, and abbreviating $F_{\kappa\lambda\mu\nu} \equiv D_\kappa D_\lambda F_{\mu\nu}$ etc.):

$$
\begin{aligned}
O_1 &= V \\
O_2 &= V^2 + \frac{1}{6} F_{\kappa\lambda} F_{\lambda\kappa} \\
O_3 &= V^3 + \frac{1}{2} V_\kappa V_\kappa + \frac{1}{2} V F_{\kappa\lambda} F_{\lambda\kappa} - \frac{2}{15}\, i\, F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} + \frac{1}{20} F_{\kappa\lambda\mu} F_{\kappa\mu\lambda} \\
O_4 &= V^4 + 2 V V_\kappa V_\kappa + \frac{1}{5} V_{\kappa\lambda} V_{\lambda\kappa} + \frac{3}{5} V^2 F_{\kappa\lambda} F_{\lambda\kappa} + \frac{2}{5} V F_{\kappa\lambda} V F_{\lambda\kappa} \\
&- \frac{4}{5}\, i\, F_{\kappa\lambda} V_\lambda V_\kappa - \frac{8}{15}\, i\, V F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\kappa} + \frac{1}{5} V F_{\kappa\lambda\mu} F_{\kappa\mu\lambda} - \frac{2}{15} F_{\kappa\lambda} F_{\lambda\mu} V_{\mu\kappa} \\
&+ \frac{1}{3} F_{\kappa\lambda} F_{\mu\lambda\kappa} V_\mu + \frac{1}{3} F_{\kappa\lambda} V_\mu F_{\mu\lambda\kappa} + \frac{2}{35} F_{\kappa\lambda} F_{\lambda\kappa} F_{\mu\nu} F_{\nu\mu} + \frac{4}{35} F_{\kappa\lambda} F_{\lambda\mu} F_{\kappa\nu} F_{\nu\mu} \\
&- \frac{1}{21} F_{\kappa\lambda} F_{\lambda\mu} F_{\mu\nu} F_{\nu\kappa} - \frac{8}{105}\, i\, F_{\kappa\lambda} F_{\lambda\mu\nu} F_{\kappa\nu\mu} - \frac{6}{35}\, i\, F_{\kappa\lambda} F_{\mu\lambda\nu} F_{\mu\nu\kappa} \\
&+ \frac{11}{420} F_{\kappa\lambda} F_{\mu\nu} F_{\lambda\kappa} F_{\nu\mu} + \frac{1}{70} F_{\kappa\lambda\mu\nu} F_{\lambda\kappa\nu\mu}
\end{aligned}
$$

With the present method, a complete calculation of all coefficients was achieved to order $O(T^6)$ in the general case [22], and to order $O(T^{12})$ in the case with only a scalar field [106,143,107].

With conventional methods this expansion was previously obtained to order $\mathrm{O}(T^5)$ in the general case [206], and only recently to order $\mathrm{O}(T^7)$ in the scalar case [214]. Detailed comparisons with other methods [215,201,206] of calculating the higher derivative expansion have been made in [106,107]. We consider here only the Onofri-Zuk method, which is the one most closely related to the worldline technique.

In Onofri's work [216], the Baker-Campbell-Hausdorff formula was employed to represent the coefficients for the pure scalar case by Feynman diagrams in a one-dimensional auxiliary field theory. Those Feynman diagrams are calculated using the Green's function

$$
G^0(\tau_1,\tau_2) = \mid \tau_1 - \tau_2 \mid - (\tau_1 + \tau_2) + \frac{2}{T}\tau_1\tau_2 \tag{7.11}
$$

which is the kernel for the second derivative operator on an interval of length T appropriate to the boundary conditions

$$
x(0) = x(T) = 0 \tag{7.12}
$$

[22] See the recent [213] for an application of this result to the approximate calculation of the one-loop contribution by massive quarks to the QCD vacuum tunneling amplitude.

This representation was then used by Zuk to calculate the effective Lagrangian for the pure scalar case up to the terms with four derivatives [199,200]. This author further generalized the method to the gauge field case, and also used Fock-Schwinger gauge to enforce manifest gauge invariance [217]. The same Green's function is used in the "Quantum Mechanical Path Integral Method" [218,82,219,220], which may be considered as an extension of the Onofri-Zuk formalism.

To see the connection to our formalism, first note that the Green's function which we used for the evaluation of the reduced path integral $\int \mathcal{D}y(\tau)$, eq.(1.16), is by no means unique. Since the naive defining equation

$$\frac{1}{2}\frac{\partial^2}{\partial\tau_1^2}G(\tau_1,\tau_2) = \delta(\tau_1-\tau_2) \tag{7.13}$$

has no periodic solutions, it needs to be modified by the introduction of a background charge, leading to [64]

$$\frac{1}{2}\frac{\partial^2}{\partial\tau_1^2}G^{\rho}(\tau_1,\tau_2) = \delta(\tau_1-\tau_2) - \rho(\tau_1) \tag{7.14}$$

The distribution of the background charge ρ along the circle is arbitrary, except that it should integrate to unity,

$$\int_0^T d\tau\rho(\tau) = 1 \tag{7.15}$$

That this is necessary can be seen by integrating eq.(7.14) in the first variable.

If one further requires the Green's function G^{ρ} to be symmetric in its both arguments, periodicity determines it up to an irrelevant constant. The solution can be expressed in terms of the standard Green's function G_B, eq.(1.16), as follows,

$$\begin{aligned} G^{\rho}(\tau_1,\tau_2) \quad &= \quad G_B(\tau_1,\tau_2) - \int_0^T d\sigma\rho(\sigma)G_B(\sigma,\tau_2) - \int_0^T d\sigma G_B(\tau_1,\sigma)\rho(\sigma) \\ &\quad + \int_0^T d\sigma_1 \int_0^T d\sigma_2\, \rho(\sigma_1)G_B(\sigma_1,\sigma_2)\rho(\sigma_2) \end{aligned} \tag{7.16}$$

Any such G^{ρ} can be used as a Green's function for the evaluation of the path integral eq. (1.8). Different choices of ρ must lead to the same effective action or scattering amplitude.

This is easily verified by the following little argument well-known from string perturbation theory. Let us consider the scalar field theory case first. For the ϕ^3 scattering amplitude, eq.(4.14) becomes, if a general ρ is used,

$$\begin{aligned} \Gamma[p_1,\dots,p_N] \quad &= \quad \frac{1}{2}(-\lambda)^N(2\pi)^D\delta(\sum p_i)\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}}e^{-m^2T} \\ &\quad \times \prod_{i=1}^N \int_0^T d\tau_i \exp\Bigl[\frac{1}{2}\sum_{i,j=1}^N G^{\rho}(\tau_i,\tau_j)p_i\cdot p_j\Bigr] \end{aligned} \tag{7.17}$$

Using eq.(7.16) and momentum conservation it is immediately seen that all ρ - dependence drops out in the exponent.

For the effective action the equivalence works in almost the same way. If only V is present, eq.(7.3) after Wick contraction turns into

$$\begin{aligned}\Gamma_{\rm scal}[V] &= \int_0^\infty \frac{dT}{T}(4\pi T)^{-D/2}{\rm e}^{-m^2T}{\rm tr}\int d^Dx_0\sum_{n=0}^{\infty}\frac{(-1)^n}{n}T\int_0^T d\tau_1\int_0^{\tau_1}d\tau_2\ ...\ \int_0^{\tau_{n-2}}d\tau_{n-1}\\ &\quad\times\exp\Bigl[-\frac{1}{2}\sum_{i,j=1}^n G^\rho(\tau_i,\tau_j)\partial_{(i)}\cdot\partial_{(j)}\Bigr]V^{(1)}(x_0)...V^{(n)}(x_0)\end{aligned} \tag{7.18}$$

Using eq.(7.16) we may rewrite the exponent as follows,

$$\begin{aligned}-\frac{1}{2}\sum_{i,j=1}^n G^{(\rho)}(\tau_i,\tau_j)\partial_{(i)}\cdot\partial_{(j)} &= -\frac{1}{2}\sum_{i,j=1}^n G_B(\tau_i,\tau_j)\partial_{(i)}\cdot\partial_{(j)}\\ &\quad+\int_0^T d\sigma\rho(\sigma)\Bigl(\sum_{i=1}^n\partial_{(i)}\cdot\sum_{j=1}^n G_B(\sigma,\tau_j)\partial_{(j)}\Bigr)\\ &\quad-\frac{1}{2}\int_0^T d\sigma_1\int_0^T d\sigma_2\,\rho(\sigma_1)G_B(\sigma_1,\sigma_2)\rho(\sigma_2)\sum_{i,j=1}^n\partial_{(i)}\cdot\partial_{(j)}\end{aligned} \tag{7.19}$$

This shows that all ρ – dependent terms in the effective Lagrangian carry at least one factor of $\sum_{i=1}^n\partial_{(i)}$. They are thus total derivative terms and will disappear in the final x_0 – integration (under appropriate boundedness conditions on the background field at infinity). The same applies to the dependence on a constant which one could always add to G^ρ.

This argument easily carries over to the gauge theory case, if one uses the Bern-Kosower master formula eq.(1.18) and its effective action analogue. Different admissible Green's functions will thus in general produce different effective Lagrangians, but the same effective action and scattering amplitudes.

However, from string theory it is also known that the choice of the background charge can have some technical significance at intermediate stages of calculations [22,221]. The Green's function G_B usually used in the string–inspired approach corresponds to the choice of a constant ρ,

$$\rho(\tau)=\frac{1}{T} \tag{7.20}$$

This is the only choice leading to a translation-invariant Green's function. If one chooses the function

$$\rho(\tau)=\delta(\tau) \tag{7.21}$$

instead, eq.(7.16) yields just the one used by Onofri, eq.(7.11). Expansion of the path integral as in eq.(7.9) then precisely generates Zuk's one-dimensional Feynman rules. The final result

after integrating out x_0 will be the same. However, the difference in the choice of ρ turns out to have some nontrivial technical consequences:

With our choice of the worldline Green's function partial integrations never become necessary in the reduction process. This is not true for the Onofri-Zuk approach, a fact which becomes particularly conspicuous in the pure scalar case [106]. Here the effective Lagrangian resulting from our method is already minimal after identification of cyclically equivalent terms, while large numbers of partial integrations turn out to be necessary to further minimize Zuk's result.

Moreover, due to the translational invariance of the worldline Green's function eq.(1.16) cyclically equivalent terms always come with the same numerical coefficient. This considerably facilitates the cyclic identification process. Again, this property does not hold true if one uses the Green's function eq. (7.11); for example, of the three cyclically equivalent terms $V_\mu V_\mu V$, $VV_\mu V_\mu$ and $V_\mu V V_\mu$ appearing in the scalar effective action at $\mathrm{O}(T^4)$ the first two then get assigned the same coefficient, while the coefficient of the third one is different.

Finally, let us to note that the ambiguity in the choice of ρ has an interpretation already at the path integral level. From eq.(7.16) it can be read off that G^ρ fulfills

$$\int_0^T d\sigma \rho(\sigma) G^\rho(\sigma,\tau_2) = \int_0^T d\sigma G^\rho(\tau_1,\sigma)\rho(\sigma) = 0 \tag{7.22}$$

(for any τ_1, τ_2). This indicates that a given background charge ρ corresponds to the following generalization of the zero mode fixing eqs.(4.4),

$$\begin{aligned}
\int \mathcal{D}x &= \int dx_0 \int \mathcal{D}y \\
x^\mu(\tau) &= x_0^\mu + y^\mu(\tau) \\
\int_0^T d\tau \rho(\tau) y^\mu(\tau) &= 0
\end{aligned} \tag{7.23}$$

In particular, for $\rho(\tau) = \delta(\tau)$ one recovers the boundary conditions eq.(7.12), $y(0) = y(T) = 0$.

In the "string-inspired" formalism, the effective Lagrangian $\mathcal{L}(x_0)$ is obtained as a path integral over the space of all loops having x_0 as their common center of mass; in the Onofri-Zuk formalism, as a path integral over the space of all loops intersecting in x_0. And indeed, for Onofri's original formalism precisely this path integral representation was already provided by Fujiwara et al. [222]. Clearly the center of mass choice is more "symmetric", so that it is intuitively reasonable that it should lead to a more compact form for the effective Lagrangian.

7.2. Other Backgrounds

The above procedure can be extended to the case of a mixed vector – axialvector background without difficulties; see [105] for the computation of some heat kernel coefficients for this background along the above lines. To the contrary, the inclusion of gravitational backgrounds poses new and interesting conceptual problems. Here the problems connected to the existence of different ordering prescriptions for the quantum mechanical Hamiltonian, which we already briefly encountered in our discussion of the gluon loop, are of a more serious nature. Classically,

the worldline Lagrangian for a scalar point particle coupled to a background gravitational field could be taken as

$$L^{\rm cl} = \frac{1}{2}\dot{x}^\mu g_{\mu\nu}(x)\dot{x}^\nu \tag{7.24}$$

Quantum mechanically, the ambiguity in the factor ordering of the Hamiltonian leads to the possible appearance of further terms in the path integral action, which are of order $\hbar^2$. According to the theorem by Sato [138] mentioned earlier our naive way of evaluating Gaussian path integrals requires the use of the Weyl-ordered Hamiltonian, which in turn is equivalent to using the mid-point rule in the standard time-slicing definition of the path integral [137,138,139,140,141]. This procedure leads to the following "quantum" Lagrangian [140],

$$L^{\rm qu}_{\rm TS} = \frac{1}{2}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu + \frac{\hbar^2}{8}\Bigl(R + g^{\mu\nu}\Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\nu\kappa}\Bigr) \tag{7.25}$$

where $\Gamma^\kappa_{\mu\lambda}$ denotes the Christoffel symbol.

Moreover, in a curved background the path integral measure also becomes nontrivial. Since, as usual, we wish to evaluate the path integral in terms of one-dimensional Feynman diagrams, it is natural to absorb this measure into the action by the introduction of appropriate ghost terms into the action, in the spirit of Lee and Yang [223]. In the present context this can be done in two slightly different ways [61,224].

To further complicate matters, a careful analysis performed over the past few years by van Nieuwenhuizen, Bastianelli, and their coworkers [60,61,62,225,144,226] has established that the coefficients of the above $\hbar^2$-terms have no absolute meaning; they depend on the choice of the regularization prescription which is implicit in the definition of the path integral. For example, another plausible definition would be to expand all worldline fields, including the ghosts, in a sine expansion about the classical trajectories, and then integrate over the Fourier coefficients. Regulating the resulting expressions by a universal cutoff on these Fourier mode sums one arrives at the so-called "mode regularization". The explicit computation [225,226] shows that, if one wishes to reproduce in this scheme the known results for the heat kernel in curved space, then the above Lagrangian must be replaced by

$$L^{\rm qu}_{\rm MR} = \frac{1}{2}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu + \frac{\hbar^2}{8}\Bigl(R - \frac{1}{3}g^{\mu\nu}g^{\kappa\lambda}g_{\rho\sigma}\Gamma^\rho_{\mu\kappa}\Gamma^\sigma_{\nu\lambda}\Bigr) \tag{7.26}$$

This ambiguity has also a natural interpretation in terms of the one-dimensional quantum field theory defined by the path integral. This field theory constitutes a super-renormalizable nonlinear sigma model, which by power counting has superficial ultraviolet divergences at the one- and two-loop levels, but not at higher loop orders [23]. Those ultraviolet divergences cancel out in the sum of terms, but, as is usual in such cases, leave a finite ambiguity embodied by the above $\hbar^2$ - terms. Their coefficients cannot be determined inside the one-dimensional field theory without further input. Requesting the result of the worldline perturbation series to reproduce the usual heat kernel expansion for the space-time field theory provides such an input, and suffices to fix them completely. Once this has been done, the coefficient of the R - term turns out to be universal for the regularization schemes considered in the works quoted above. Since the heat kernel is a covariant quantity, the appearance of the other, noncovariant

[23] Here the loop counting refers to the one-dimensional worldline field theory; in terms of the four-dimensional field theory our discussion is, of course, at the one-loop level.

terms in the worldline Lagrangians (7.25),(7.26) is clearly connected to the fact that both regularizations used, time-slicing and mode regularization, break the covariance; the role of the explicit noncovariant terms in the Lagrangians is to compensate for this. This leads to the question whether some regularization can be found which would avoid this covariance breaking. Very recently, Kleinert and Chervyakov [227,228,229] have claimed that one-dimensional dimensional regularization provides such a scheme, and they verified the absence of any $\hbar^2$ - terms in a simpler model with a one-dimensional target space. In [230,231] it was then shown that, in the four-dimensional case, this scheme is indeed free of noncovariant counterterms, although the R - term is still necessary:

$$L_{\rm DR}^{\rm qu} = \frac{1}{2}\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu + \frac{\hbar^2}{8}R \tag{7.27}$$

This scheme therefore seems to be the most promising one for future applications of the string-inspired formalism to curved-space calculations [24] .

To use any of these worldline actions for the calculation of the higher derivative expansion of the gravitational effective action, one would now wish to choose a Riemann normal coordinate system centered at the loop center of mass x_0. This is the gravitational analogue of Fock-Schwinger gauge, and allows one to rewrite the Taylor expansion of the metric at x_0 in terms of covariant derivatives of the curvature tensor [232,233],

$$g_{\mu\nu}(x) = \delta_{\mu\nu} + \frac{1}{3}y^\alpha y^\beta R_{\mu\alpha\beta\nu} + \frac{1}{6}y^\gamma y^\alpha y^\beta \nabla_\gamma R_{\mu\alpha\beta\nu} + \cdots \tag{7.28}$$

Individual terms in the higher derivative expansion can then again be calculated by the application of appropriate Wick contraction rules [60,61,224,62,225,144,226,185].

However, here another rather subtle complication appears in the choice of the worldline propagator. As we remarked in the previous section for the gauge theory case, and explicitly demonstrated for scalar field theory, there exists a large family of admissible worldline propagators, and the effective actions computed with different such propagators differ by total derivative terms. While this statement remains true in the curved space case, an explicit computation at the two-loop level has revealed [144] that those total derivative terms are in general *not covariant*. If one wishes to reproduce precisely the standard heat kernel expansion, which is manifestly covariant, then the Onofri-Zuk propagator (7.11) must be used, corresponding to the boundary conditions $y(0) = y(T) = 0$ on the path integral after the zero mode fixing. The use of the standard worldline Green's function G_B, on the other hand, leads to a result which differs from this by noncovariant total derivative terms. This poses no problems in principle, but in practice, since it invalidates the application of Riemann normal coordinates, which are useful only for the computation of covariant quantities [25]. Thus the present state of affairs is that, in the application of the worldline technique to curved space effective actions, one either has to forgo the convenience of using the translation invariant worldline propagator G_B or, worse, of the use of Riemann normal coordinates [26] .

[24] As a necessary preliminary step to any such application in [231] it was shown how to extend dimensional regularization to a compact time intervall.

[25] To be precise, the application of such coordinates to the computation of a noncovariant quantitity will produce an *apparently* covariant result which is correct in that particular coordinate system but not in others.

[26] It should be noted that, a priori, the analogous problem could have appeared also in the gauge theory case in the form of non – gauge invariant total derivative terms. Our use of Fock-Schwinger coordinates in the previous section was possible only because the compatibility of the standard propagator G_B with gauge invariance is known on general grounds (see chapter 4).

8. Multiloop Worldline Green's Functions

While one could clearly construct multiloop formulas of the Bern-Kosower type starting with formulas such as eqs.(4.14),(1.18),(4.32), and then sewing together pairs of external legs, such a procedure turns out to be unnecessarily cumbersome. In the spirit of the Bern-Kosower formalism, we would rather like to completely avoid the appearance of internal momenta. This is not only for aesthetic reasons; it is the absence of internal momenta which, in the Bern-Kosower formalism, reduces the number of independent kinematic invariants from the very beginning, thereby rendering the spinor helicity formalism even more useful than usual.

We will rather follow the example of string theory, where multiloop amplitudes can be represented as path integrals over Riemann surfaces of higher genus, embedded into some target space. The Green's function of the Laplacian (or some other kinetic operator) for those surfaces [5,22] is then the basic quantity needed for their calculation. In the infinite string tension limit, the Riemann surfaces correspond to graphs. If we wish to preserve the analogy with string theory, we ought to find out how to calculate path integrals over graphs embedded into spacetime.

This is by no means a new idea. For the example of ϕ^3 – theory, it has been repeatedly pointed out that it should be possible to construct multiloop amplitudes in terms of path integrals over graphs [234,235,121,236,237]. The proper-time lengths of the propagators making up those graphs would then just correspond to the moduli parameters in string theory. However, none of those authors provided an explicit computational prescription for the evaluation of this kind of path integral. In [84], M.G. Schmidt and the present author proposed such a prescription, based on the concept of Green's functions defined on graphs. Let us therefore begin with explaining how to construct such "multiloop worldline Green's functions" at the two-loop level.

8.1. The 2-Loop Case

At first glance, this looks like an ill-defined problem. In contrast to the circle, a general graph is not a differentiable manifold, and it is a priori not obvious how to define the second derivative operator at the node points.

Instead, we will pose the following simple question. How does the Green's function $G_B(\tau_1, \tau_2)$ between two fixed points τ_1, τ_2 on the circle change, if we insert, between two other points τ_a and τ_b, a (scalar) propagator of fixed proper-time length $\bar{T}$ (fig. 19)?

To answer this question, let us start with the worldline-path integral representation for the one-loop two-point – amplitude in ϕ^3 – theory, and sew together the two external legs. The result is, of course, the vacuum path integral with a propagator insertion:

$$\Gamma^{(2)}_{\rm vac} = \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} \int \mathcal{D}x \int_0^T d\tau_a \int_0^T d\tau_b \, \langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle \, \exp\Bigl[-\int_0^T d\tau \frac{\dot{x}^2}{4}\Bigr] \tag{8.1}$$

Here $\langle \phi(x(\tau_a))\phi(x(\tau_b))\rangle$ is the x–space scalar propagator in D dimensions, which, if we specialize to the massless case for a moment, would read

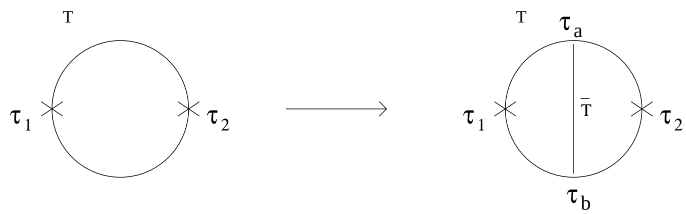


Figure 19: Change of the one-loop Green's function by a propagator insertion.

$$\langle\phi(x(\tau_a))\phi(x(\tau_b))\rangle = \frac{\Gamma(\frac{D}{2}-1)}{4\pi^{\frac{D}{2}}\Big[(x_a-x_b)^2\Big]^{(\frac{D}{2}-1)}} \tag{8.2}$$

Clearly this form of the propagator is not suitable for calculations in our auxiliary one-dimensional field theory. The approach based on worldline Green's functions which we have in mind will work nicely only if we can manipulate all our path integrals into Gaussian form. To obtain a Gaussian path integral, we therefore make further use of the Schwinger proper-time representation to exponentiate the propagator insertion,

$$\langle\phi(x(\tau_a))\phi(x(\tau_b))\rangle = \int_0^\infty d\bar{T} e^{-m^2\bar{T}}(4\pi\bar{T})^{-\frac{D}{2}}\exp\Big[-\frac{(x(\tau_a)-x(\tau_b))^2}{4\bar{T}}\Big] \tag{8.3}$$

where the propagator mass was also reinstated. We have then the following path integral representation of the two – loop vacuum amplitude:

$$\begin{aligned}\Gamma^{(2)}_{\rm vac} \quad = \quad & \int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}\int_0^\infty d\bar{T}(4\pi\bar{T})^{-\frac{D}{2}}e^{-m^2\bar{T}}\int_0^T d\tau_a\int_0^T d\tau_b \\ & \times\int\mathcal{D}x\,\exp\Big[-\int_0^T d\tau\,\frac{\dot{x}^2}{4}-\frac{(x(\tau_a)-x(\tau_b))^2}{4\bar{T}}\Big]\end{aligned} \tag{8.4}$$

The propagator insertion has, for fixed parameters $\bar{T},\tau_a,\tau_b$, just produced an additional contribution to the original free worldline action. Moreover, this term is quadratic in x, so that we can hope to absorb it into the free worldline Green's function. For this purpose, it is useful to introduce an integral operator B_{ab} with integral kernel

$$B_{ab}(\tau_1,\tau_2) = \Big[\delta(\tau_1-\tau_a)-\delta(\tau_1-\tau_b)\Big]\Big[\delta(\tau_a-\tau_2)-\delta(\tau_b-\tau_2)\Big] \tag{8.5}$$

(B_{ab} acts trivially on Lorentz indices). We may then rewrite

$$(x(\tau_a) - x(\tau_b))^2 = \int_0^T d\tau_1 \int_0^T d\tau_2 \, x(\tau_1) B_{ab}(\tau_1, \tau_2) x(\tau_2) \tag{8.6}$$

Obviously, the presence of the additional term corresponds to changing the defining equation for G_B, eq.(1.15), to

$$G_B^{(1)}(\tau_1, \tau_2) = 2\langle \tau_1 \mid \Bigl(\frac{d^2}{d\tau^2} - \frac{B_{ab}}{\bar{T}}\Bigr)^{-1} \mid \tau_2 \rangle \tag{8.7}$$

After eliminating the zero-mode as before, this modified propagator can be constructed simply as a geometric series:

$$\Bigl(\frac{d^2}{d\tau^2} - \frac{B_{ab}}{\bar{T}}\Bigr)^{-1} = \Bigl(\frac{d}{d\tau}\Bigr)^{-2} + \Bigl(\frac{d}{d\tau}\Bigr)^{-2} \frac{B_{ab}}{\bar{T}} \Bigl(\frac{d}{d\tau}\Bigr)^{-2} + \Bigl(\frac{d}{d\tau}\Bigr)^{-2} \frac{B_{ab}}{\bar{T}} \Bigl(\frac{d}{d\tau}\Bigr)^{-2} \frac{B_{ab}}{\bar{T}} \Bigl(\frac{d}{d\tau}\Bigr)^{-2} + \cdots \tag{8.8}$$

Noting that

$$B_{ab}\Bigl(\frac{d}{d\tau}\Bigr)^{-2} B_{ab} = -G_{Bab} B_{ab} \tag{8.9}$$

we can explicitly sum this series, and obtain [84]

$$G_B^{(1)}(\tau_1, \tau_2) = G_B(\tau_1, \tau_2) + \frac{1}{2} \frac{[G_B(\tau_1, \tau_a) - G_B(\tau_1, \tau_b)][G_B(\tau_a, \tau_2) - G_B(\tau_b, \tau_2)]}{\bar{T} + G_B(\tau_a, \tau_b)} \tag{8.10}$$

The worldline Green's function between points τ_1 and τ_2 is thus simply the one-loop Green's function plus one additional piece, which takes the effect of the insertion into account. Observe that this piece can still be written in terms of the various one-loop Green's functions G_{Bij}. However it is not a function of $\tau_1 - \tau_2$ any more, nor is its coincidence limit a constant (for alternative derivations of this expression see [84,91,86]).

Knowledge of this Green's function is not yet sufficient for performing two-loop calculations. We also need to know how the path integral determinant is changed by the propagator insertion. Using the $ln\, det = tr\, ln$ – formula this can be easily calculated, and yields

$$\frac{\int \mathcal{D}y \, \exp\Bigl[-\int_0^T d\tau \frac{\dot{y}^2}{4} - \frac{(y(\tau_a)-y(\tau_b))^2}{4\bar{T}}\Bigr]}{\int \mathcal{D}y \, \exp\Bigl[-\int_0^T d\tau \frac{\dot{y}^2}{4}\Bigr]} = \frac{\mathrm{Det}'_{\mathrm{P}}\bigl(\frac{d^2}{d\tau^2} - \frac{B_{ab}}{\bar{T}}\bigr)^{-\frac{D}{2}}}{\mathrm{Det}'_{\mathrm{P}}\bigl(\frac{d^2}{d\tau^2}\bigr)^{-\frac{D}{2}}} = \Bigl(1 + \frac{G_{Bab}}{\bar{T}}\Bigr)^{-\frac{D}{2}} \tag{8.11}$$

To summarize, the insertion of a scalar propagator into a scalar loop can, for fixed values of the proper-time parameters, be completely taken into account by changing the path integral normalization, and replacing G_B by $G_B^{(1)}$. The vertex operators remain unchanged.

In this way we arrive at the following two-loop generalization of eq. (4.14),

$$\int_0^\infty \frac{dT}{T} \int_0^\infty d\bar{T} e^{-m^2(T+\bar{T})} (4\pi)^{-D} \int_0^T d\tau_a \int_0^T d\tau_b \, [T\bar{T} + TG_B(\tau_a, \tau_b)]^{-\frac{D}{2}}$$
$$\times \prod_{i=1}^N \int_0^T d\tau_i \, \exp\Bigl[\sum_{k,l=1}^N G_B^{(1)}(\tau_k, \tau_l) k_k \cdot k_l\Bigr] \tag{8.12}$$

For fixed N, this integral represents a certain linear combination of two-loop diagrams in ϕ^3 – theory which have N legs on the loop, and no leg on the internal line (fig. 20).

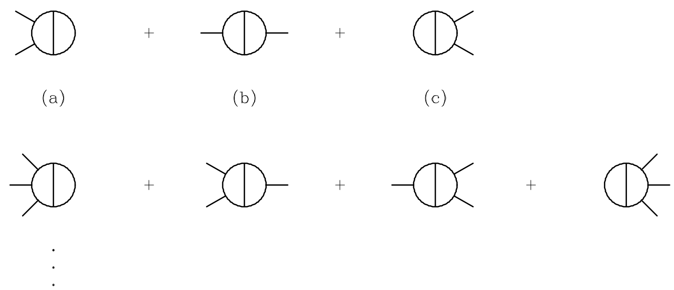


Figure 20: Summation of diagrams with N legs on the loop.

Observe that we have diagonal terms in the exponential, since $G_B^{(1)}$ has $G_B^{(1)}(\tau,\tau) \neq 0$ in general. Self-contractions of vertex operators must therefore be taken into account. As always, momentum conservation allows one to absorb the diagonal terms into the non-diagonal ones, however in contrast to our previous experiences the coincidence limit of $G_B^{(1)}$ is not constant. To ensure that the subtracted Green's function has a zero coincidence limit, it must now be defined in the following way [27]

$$\bar{G}_B^{(1)}(\tau_1,\tau_2) \equiv G_B^{(1)}(\tau_1,\tau_2) - \frac{1}{2}G_B^{(1)}(\tau_1,\tau_1) - \frac{1}{2}G_B^{(1)}(\tau_2,\tau_2) \tag{8.13}$$

Explicitly this gives

$$\bar{G}_B^{(1)}(\tau_1,\tau_2) \quad = \quad G_{B12} - \frac{1}{4}\frac{\Bigl(G_{B1a} - G_{B1b} - G_{B2a} + G_{B2b}\Bigr)^2}{\bar{T} + G_{Bab}} \tag{8.14}$$

We note the following properties of this "subtracted two-loop worldline Green's function":

1. As one would expect $\bar{G}_B^{(1)}$ reduces to G_B in the limit where the proper-time $\bar{T}$ of the inserted propagator becomes infinite.

2. For the case that both points $\tau_{1,2}$ are located on the same side of the propagator insertion we can rewrite $\bar{G}_B^{(1)}$ as the one-loop Green's function G_B with a modified global proper-time $T \to T'$. In the parametrization of fig. (21b) this new proper-time is given by $T' = T_1 + \frac{T_2 T_3}{T_2 + T_3}$ [28].

[27] We remark that an analogous ambiguity appears in the electric circuit approach to Feynman parameter integration [80,81,79]. In the terminology of [79] the condition $\bar{G}_B^{(1)}(\tau,\tau) = 0$ corresponds to the choice of a "level zero scheme".

[28] This property also has a direct analogue in the electric circuit formalism.

3. $\bar{G}_B^{(1)}$ is *not* translation invariant, i.e. the equation

$$\Big(\frac{\partial}{\partial\tau_1}+\frac{\partial}{\partial\tau_2}\Big)\bar{G}_B^{(1)}(\tau_1,\tau_2) \quad = \quad 0 \tag{8.15}$$

is not true in general. It *does* hold, however, for $\tau_{1,2}$ on the same side of the propagator insertion, as follows directly from the previous property.

So far we are restricted to inserting vertex operators on the loop only. Due to the symmetry of the diagram, this restriction is easily removed. Obviously, for any two points on the two-loop vacuum graph we may regard those to be on the loop, and the remaining branch – or one of the remaining branches – to be the inserted line. We can thus always use our formula eq.(8.10) up to a reparametrization. In contrast to the string-theoretic worldsheet Green's function the worldline Green's function is a "bi-scalar", i.e. it transforms trivially under reparametrizations.

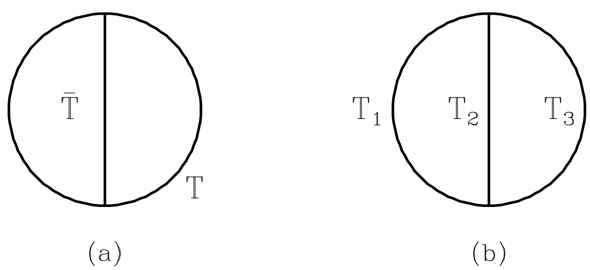


Figure 21: Two different parametrizations of the two-loop diagram.

We therefore now find it convenient to switch to the parametrization of fig. 21b, which is symmetric with regard to the three branches. We have then three "moduli parameters" T_1, T_2 and T_3, and the location of a vertex operator on branch i will be denoted by a parameter τ_i running from 0 to T_i.

If we now fix the number of external legs on branch i to be n_i, carrying momenta $k_i^{(i)},\dots,k_{n_i}^{(i)}$, we obtain the following obvious generalization of eq. (8.12):

$$\begin{aligned}
&\prod_{a=1}^{3}\int_0^\infty dT_a e^{-m^2(T_1+T_2+T_3)}(4\pi)^{-D}(T_1T_2+T_1T_3+T_2T_3)^{-\frac{D}{2}}\\
&\times\prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\prod_{m=1}^{n_3}\int_0^{T_1}d\tau_i^{(1)}\int_0^{T_2}d\tau_j^{(2)}\int_0^{T_3}d\tau_k^{(3)}\\
&\times\exp\Biggl[\sum_{r<s}\sum_{k=1}^{n_r}\sum_{l=1}^{n_s}G_{Brs}^{\rm sym}(\tau_k^{(r)},\tau_l^{(s)})k_k^{(r)}\cdot k_l^{(s)}+\sum_{r=1}^{3}\frac{1}{2}\sum_{k,l=1}^{n_r}G_{Brr}^{\rm sym}(\tau_k^{(r)},\tau_l^{(r)})k_k^{(r)}\cdot k_l^{(r)}\Biggr]
\end{aligned} \tag{8.16}$$

Here the $G_{B11}^{\rm sym}, G_{B33}^{\rm sym}, G_{B13}^{\rm sym}$ are related to $G_B^{(1)}$ by the mentioned reparametrization. Again it is convenient to absorb the diagonal coincidence terms from the beginning via eq.(8.13). After this subtraction, the two-loop worldline Green's function in symmetric parametrization becomes

$$
\begin{aligned}
\bar{G}^{\rm sym}_{B11}(\tau_1^{(1)},\tau_2^{(1)}) &= \Delta \mid \tau_1^{(1)}-\tau_2^{(1)} \mid \Big[(T_1 - \mid \tau_1^{(1)}-\tau_2^{(1)} \mid)(T_2+T_3)+T_2T_3\Big] \\
&= \mid \tau_1^{(1)}-\tau_2^{(1)} \mid - \Delta(T_2+T_3)(\tau_1^{(1)}-\tau_2^{(1)})^2 \\
\bar{G}^{\rm sym}_{B12}(\tau^{(1)},\tau^{(2)}) &= \Delta\Big[T_3(\tau^{(1)}+\tau^{(2)})[T_1+T_2-(\tau^{(1)}+\tau^{(2)})] \\
&\quad +\tau^{(2)}(T_2-\tau^{(2)})T_1+\tau^{(1)}(T_1-\tau^{(1)})T_2\Big] \\
&= \tau^{(1)}+\tau^{(2)} - \Delta\Big[{\tau^{(1)}}^2T_2+{\tau^{(2)}}^2T_1+(\tau^{(1)}+\tau^{(2)})^2T_3\Big] \\
\Delta &= (T_1T_2+T_1T_3+T_2T_3)^{-1}
\end{aligned}
\tag{8.17}
$$

plus permuted $\bar{G}_{Bij}$'s.

8.2. Comparison with Feynman Diagrams

Let us look more closely at the two-point case, and compare our approach with the corresponding Feynman diagram calculation. For $N=2$, eq.(8.12) reads

$$
\begin{aligned}
&\int_0^\infty \frac{dT}{T}\int_0^\infty d\bar{T} e^{-m^2(T+\bar{T})}(4\pi)^{-D}\int_0^T d\tau_a \int_0^T d\tau_b \,[T\bar{T}+TG_B(\tau_a,\tau_b)]^{-\frac{D}{2}} \\
&\times \int_0^T d\tau_1 \int_0^T d\tau_2 \exp\Big[\Big(\frac{1}{2}G_B^{(1)}(\tau_1,\tau_1)+\frac{1}{2}G_B^{(1)}(\tau_2,\tau_2)-G_B^{(1)}(\tau_1,\tau_2)\Big)k^2\Big]
\end{aligned}
\tag{8.18}
$$

This should correspond to the sum of graphs (a), (b) and (c) of fig. 20. A straightforward Feynman parameter calculation of graph (a) results in ($k=k_1=-k_2$) [29]

$$
\int_0^\infty d\hat{T}\,(4\pi)^{-D}e^{-m^2\hat{T}}\prod_{i=1}^5\int d\alpha_i\,\delta(\hat{T}-\sum_{i=1}^5\alpha_i)\,[P^{(a)}(\alpha_i)]^{-\frac{D}{2}}\exp[-Q^{(a)}(\alpha_i)k^2]
\tag{8.19}
$$

with

$$
\begin{aligned}
P^{(a)} &= \alpha_5(\alpha_1+\alpha_2+\alpha_3+\alpha_4)+\alpha_3(\alpha_1+\alpha_2+\alpha_4) \\
P^{(a)}Q^{(a)} &= \alpha_1[\alpha_5(\alpha_2+\alpha_3+\alpha_4)+\alpha_2\alpha_3+\alpha_3\alpha_4]
\end{aligned}
\tag{8.20}
$$

The analogue of the transformation eq. (4.15) can be directly read off fig. 22a

$$
\begin{aligned}
\alpha_1 &= \tau_1-\tau_2 \\
\alpha_2 &= T-\tau_1+\tau_b \\
\alpha_3 &= \tau_a-\tau_b \\
\alpha_4 &= \tau_2-\tau_a \\
\alpha_5 &= \bar{T}
\end{aligned}
\tag{8.21}
$$

[29] $P^{(a)}$ was misprinted in [84].

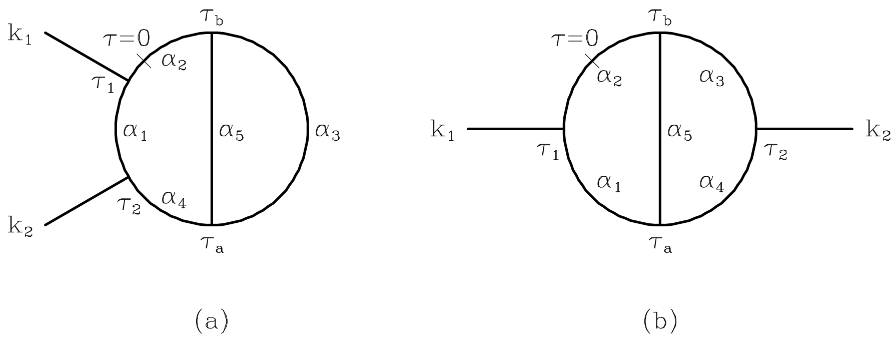


Figure 22: Reparametrization of the two-point two-loop diagrams.

As expected, it leads to the identifications

$$\begin{aligned} P^{(a)} &= T\bar{T}[1 + \frac{1}{\bar{T}} G_B(\tau_a, \tau_b)] \\ Q^{(a)} &= \bar{G}_B^{(1)}(\tau_1, \tau_2) \end{aligned} \tag{8.22}$$

The Feynman calculation for diagram (b) yields polynomials

$$\begin{aligned} P^{(b)} &= \alpha_5(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + (\alpha_1 + \alpha_2)(\alpha_3 + \alpha_4) \\ P^{(b)} Q^{(b)} &= \alpha_5(\alpha_2 + \alpha_3)(\alpha_1 + \alpha_4) + \alpha_1\alpha_2(\alpha_3 + \alpha_4) + \alpha_3\alpha_4(\alpha_1 + \alpha_2) \end{aligned} \tag{8.23}$$

which are different as functions of the variables α_i, but after the corresponding transformation

$$\begin{aligned} \alpha_1 &= \tau_1 - \tau_a \\ \alpha_2 &= \tau_b + T - \tau_1 \\ \alpha_3 &= \tau_2 - \tau_b \\ \alpha_4 &= \tau_a - \tau_2 \\ \alpha_5 &= \bar{T} \end{aligned} \tag{8.24}$$

identify with the *same* expressions (8.22). It should be noted that this becomes apparent only after everything has been expressed in terms of G_B, due to the absolute sign contained in that function.

Our worldline formula eq. (8.18) thus indeed unifies the three α – parameter integrals, arising in the calculation of diagrams (a), (b) and (c), in a single τ – parameter integral. This correspondence has also been checked for a number of diagrams with more external legs, as well as for diagrams with legs on all three branches.

This is remarkable since in field theory diagrams (a) and (b) have very different properties. Both in the massless and in the massive cases the integrals arising from topology (b) are

less elementary than the ones from (a). To understand how this comes about we need only remember a property of $\bar{G}_B^{(1)}$ stated above, namely that it is translation invariant for (a) but not for (b). Therefore for (a) the parameter integral has a redundancy, and eq.(8.15) can be used to reduce the number of integrations by one.

Quite obviously we have found here a universality property which is not visible in ordinary Feynman parameter calculations, and constitutes a field theory relic of the fact mentioned in the introduction, namely that string perturbation theory does not suffer from the usual proliferation of terms due to the existence of many different topologies.

8.3. Higher Loop Orders

The whole procedure generalizes without difficulty to the case of m propagator insertions, resulting in an integral representation combining into one expression all diagrams with N legs on the loop, and m inserted propagators:

$$\begin{aligned} &\int_0^\infty \frac{dT}{T} T^{-\frac{D}{2}} (4\pi)^{-(m+1)\frac{D}{2}} \prod_{j=1}^m \int_0^\infty d\bar{T}_j e^{-m^2(T+\sum_{j=1}^m \bar{T}_j)} \int_0^T d\tau_{a_j} \int_0^T d\tau_{b_j} \\ &\times \prod_{i=1}^N \int_0^T d\tau_i \, N^{(m)\,\frac{D}{2}} \exp\Bigl[\frac{1}{2}\sum_{k,l=1}^N G_B^{(m)}(\tau_k,\tau_l) k_k\cdot k_l\Bigr] \end{aligned} \tag{8.25}$$

where

$$\begin{aligned} N^{(m)} &= \mathrm{Det}(A^{(m)}) \\ G_B^{(m)}(\tau_1,\tau_2) &= G_B(\tau_1,\tau_2) \\ &\quad + \frac{1}{2}\sum_{k,l=1}^m [G_B(\tau_1,\tau_{a_k}) - G_B(\tau_1,\tau_{b_k})] A_{kl}^{(m)} [G_B(\tau_2,\tau_{a_l}) - G_B(\tau_2,\tau_{b_l})] \end{aligned} \tag{8.26}$$

and the symmetric $m\times m$ – matrix $A^{(m)}$ is defined by

$$\begin{aligned} A^{(m)} &= \Bigl[\bar{T} - \frac{C}{2}\Bigr]^{-1} \\ \bar{T}_{kl} &= \bar{T}_k \delta_{kl} \\ C_{kl} &= G_B(\tau_{a_k},\tau_{a_l}) - G_B(\tau_{a_k},\tau_{b_l}) - G_B(\tau_{b_k},\tau_{a_l}) + G_B(\tau_{b_k},\tau_{b_l}) \end{aligned} \tag{8.27}$$

Here $\bar{T}_1,\ldots,\bar{T}_m$ denote the proper-time lengths of the inserted propagators.

The coincidence terms can be subtracted as in the two-loop case, leading to "subtracted" Green's functions $\bar{G}_B^{(m)}$ related to the $G_B^{(m)}$ via the same eq.(8.13). Both choices lead to the same scattering amplitudes.

Of course one is free to choose the masses of the inserted propagators to be different from each other, and from the loop mass m. To give different masses to the individual field theory propagators making up the loop is also possible, albeit only if one fixes the ordering

$$\tau_{i_1} > \tau_{i_2} > \ldots \tag{8.28}$$

of the interaction points around the loop. In this case, instead of the global factor of $e^{-m^2 T}$ one has to insert one factor of

$$e^{-m_j^2(\tau_{i_j}-\tau_{i_{j+1}})} \tag{8.29}$$

for every massive propagator, where m_j denotes the mass for the propagator connecting τ_{i_j} and $\tau_{i_{j+1}}$.

Note that the formula above gives the worldline correlator only between points on the loop. Beyond the two – loop level, the construction of the correlators involving points on the inserted propagators cannot be achieved by symmetry arguments any more. This extension was studied by Roland and Sato [87,88], who obtained explicit formulas similar to eqs.(8.26),(8.27) for the Green's function between arbitrary points on the same class of graphs. This knowledge then is sufficient to write down worldline representations for all ϕ^3 graphs which have the topology of a loop with insertions. According to graph theory [238], the set of such graphs is surprisingly large. For the first few orders of perturbation theory such a loop, or "Hamiltonian circuit", can always be found; all trivalent graphs with less than 34 vertices do have this property [30]. This would, of course, do for most practical purposes, but still poses a problem in theory.

But there is a more bothersome problem, which so far we have swept under the carpet. In checking the correspondence to Feynman graph calculations, we verified that the correct integrands were produced, but left aside the global statistical and symmetry factors for the individual diagrams. As it turns out those do not work out in the case of ϕ^3 – theory. Even in the simple example analyzed above the complete τ – integral contains diagram (a) and (b) in a ratio of $2:1$, while in field theory one would have a ratio of $1:1$.

Both these problems can be solved at the same time by an appropriate reformulation of the theory at the field theory level [89]. This can be done in various ways. In $\lambda\phi^3$ – theory, the basic idea is to rewrite the generating functional

$$Z[\eta] = \int \mathcal{D}\phi(x) \exp\Bigl\{\int dx \Bigl[-\frac{1}{2}\phi(x)(-\Box + m^2)\phi(x) + \eta(x)\phi(x) - \lambda\phi^3(x)\Bigr]\Bigr\} \tag{8.30}$$

in the following way,

$$Z[\eta] = \int \mathcal{D}A(x)\mathcal{D}\phi(x)\,\delta(A-\phi)\exp\Bigl\{\int dx \Bigl[-\frac{1}{2}\phi(x)(-\Box + m^2)\phi(x) + \eta(x)\phi(x) - \lambda A(x)\phi^2(x)\Bigr]\Bigr\} \tag{8.31}$$

After a Fourier transformation of the functional δ – function one finds that the new scalar field theory obtained does not suffer from the above problems, since the interaction between ϕ and the auxiliary field A is of the Yukawa type. For this theory the whole S-matrix can be exhausted by Hamiltonian graphs, and moreover letting "legs slide around loops" generates the correct statistical factors. A similar reformulation exists for Yang-Mills theory [95]. However the practical value of this procedure has not been established yet, and we will not pursue this matter further here.

It is interesting to compare these difficulties in the correct generation of the set of all Feynman diagrams to the situation in string perturbation theory. There the apparent advantage of being

[30] This statement assumes that we disallow insertions of the trivial one-loop propagator bubble graph. (I thank D. Kreimer for pointing this out to me.)

able to write down the full amplitude "in one piece" offered by the Polyakov path integral approach may, at higher loop orders, become increasingly illusory due to the absence of a convenient global parametrization of the corresponding moduli spaces. The cell decomposition of moduli space provided by second quantized string field theory [7] may then turn into an advantage.

8.4. Connection to String Theory

Roland and Sato [87] provided a link back to string theory by analyzing the infinite string tension limit of the Green's function $G_B^{RS(m)}$ of the corresponding Riemann surface, and identifying $\bar{G}_B^{(m)}$ with the leading order term of $G_B^{RS(m)}$ in the $\frac{1}{\alpha'}$ – expansion:

$$G_B^{RS(m)}(z_1, z_2) \quad \stackrel{\alpha' \to 0}{\longrightarrow} \quad \frac{1}{\alpha'}\bar{G}_B^{(m)}(\tau_1, \tau_2) + \text{finite} \tag{8.32}$$

Note that their derivation automatically leads to the "subtracted" version of the multiloop Green's functions.

8.5. Example: 3-Loop Vacuum Amplitude

Finally, let us have a look at the simplest example of a three – loop parameter integral calculation in this formalism [239]. This is the one where the integrand consists just of the bosonic three-loop determinant factor $(N^{(2)})^{\frac{D}{2}}$ (see eq.(8.25)). In dimensional regularization it reads

$$\Gamma^{(3)}_{\text{vac}}(D) = (4\pi)^{-\frac{3}{2}D} \int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} T^{6-\frac{3}{2}D} I(D) \tag{8.33}$$

$$I(D) = \int_0^\infty d\hat{T}_1 \, d\hat{T}_2 \int_0^1 da \, db \, dc \, dd \left[(\hat{T}_1 + G_{Bab})(\hat{T}_2 + G_{Bcd}) - \frac{C^2}{4} \right]^{-\frac{D}{2}} \tag{8.34}$$

Here $\hat{T}_{1,2} = \frac{T_{1,2}}{T}$ denote the proper-time lengths of the two inserted propagators in units of T, and $C \equiv G_{Bac} - G_{Bad} - G_{Bbc} + G_{Bbd}$.

In the following we will show how to compute the $\frac{1}{\epsilon}$ - pole of this amplitude, which is the quantity needed for applications to the calculation of renormalization group functions.

In writing eq.(8.33) we have already rescaled to the unit circle, and separated off the electron proper-time integral. This integral decouples, and just yields an overall factor of

$$\int_0^\infty \frac{dT}{T} \mathrm{e}^{-m^2 T} T^{6-\frac{3}{2}D} = \Gamma(6 - \frac{3}{2}D) \, m^{3D-12} \sim -\frac{2}{3\epsilon} \tag{8.35}$$

($\epsilon = D - 4$). The nontrivial integrations are $\int_0^1 da \, db \, dc \, dd \equiv \int_{abcd}$, representing the four propagator end points moving around the loop (fig. 23).

This fourfold integral decomposes into 24 ordered sectors, of which 16 constitute the planar (P) (fig. 23a) and 8 the non-planar (NP) sector (fig. 23b). Due to the symmetry properties of the integrand, all sectors of the same topology give an equal contribution. The integrand has

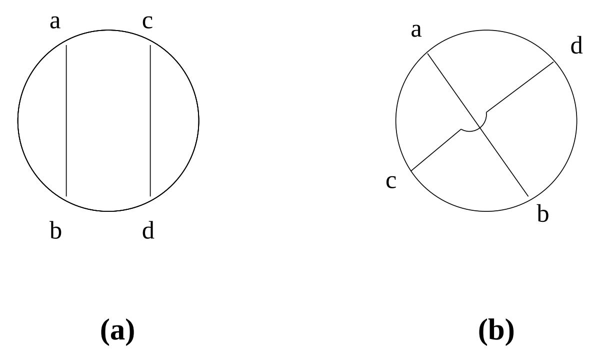


Figure 23: Planar and non-planar sectors.

a trivial invariance under the operator $\frac{\partial}{\partial a}+\frac{\partial}{\partial b}+\frac{\partial}{\partial c}+\frac{\partial}{\partial d}$, which just shifts the location of the zero on the loop.

As a first step in the calculation of $I(D)$, it is useful to add and subtract the same integral with $C=0$ and rewrite

$$I(D) = I_{\rm sing}(D) + I_{\rm reg}(D) \tag{8.36}$$

$$I_{\rm sing}(D) = \int_0^\infty d\hat{T}_1\, d\hat{T}_2 \int\limits_{abcd} \Bigl[(\hat{T}_1+G_{Bab})(\hat{T}_2+G_{Bcd})\Bigr]^{-\frac{D}{2}} \tag{8.37}$$

$I_{\rm sing}(D)$ factorizes into two identical three-parameter integrals, which are elementary:

$$I_{\rm sing}(D) = \biggl\{\int_0^\infty dT\int_0^1 du\Bigl[T+u(1-u)\Bigr]^{-\frac{D}{2}}\biggr\}^2 = \biggl[\frac{2B(2-\frac{D}{2},2-\frac{D}{2})}{D-2}\biggr]^2 \tag{8.38}$$

The point of this split is that the remainder $I_{\rm reg}(D)$ is finite. To see this, set $D=4$, expand the original integrand in $\frac{C^2}{G_{Bab}G_{Bcd}}$, and note that for all terms but the first one the zeroes of G_{Bab} (G_{Bcd}) at $a\sim b$ ($c\sim d$) are neutralized by zeroes of C^2. Since we want only the $\frac{1}{\epsilon}$ – pole of $\Gamma^{(3)}_{\rm vac}$ we can set $D=4$ in the calculation of $I_{\rm reg}(D)$. The integrations over $\hat{T}_1,\hat{T}_2$ are then elementary, and we are left with

$$I_{\rm reg}(4) \quad = \quad \int\limits_{abcd}\biggl[-\frac{4}{C^2}\ln\Bigl(1-\frac{C^2}{4G_{Bab}G_{Bcd}}\Bigr)-\frac{1}{G_{Bab}G_{Bcd}}\biggr] \tag{8.39}$$

For the calculation of this integral, observe the following simple behaviour of the function C under the operation $D_{ab}\equiv\frac{\partial}{\partial\tau_a}+\frac{\partial}{\partial\tau_b}$:

$$D_{ab}C = \pm 2\chi_{NP},\ D^2_{ab}C = 2\Bigl(\delta_{ac}-\delta_{ad}-\delta_{bc}+\delta_{bd}\Bigr) \tag{8.40}$$

where χ_{NP} denotes the characteristic function of the non-planar sector (i.e. it is zero on the planar and one on the non-planar sector). From these identities and the symmetry properties one can easily derive the following projection identities, which effectively integrate out the variable C:

$$\begin{aligned}\int_P f(C, G_{Bab}, G_{Bcd}) &= 4\int_0^1 da \int_0^a dc(a-c) f\Big(-2c(1-a), a-a^2, c-c^2\Big)\\ \int_{NP} f(C, G_{Bab}, G_{Bcd}) &= -4\int_0^1 da \int_0^a dc \int_0^{-2c(1-a)} dC f\Big(C, a-a^2, c-c^2\Big)\end{aligned} \tag{8.41}$$

Here f is an arbitrary function in the variables G, G_{Bab}, G_{Bcd}, and $\int_0^C dCf$ denotes the integral of this function in the variable C, with the other variables fixed. The integrals on the left hand side are restricted to the sectors indicated. For f the integrand of our formula eq. (8.39), we have

$$\begin{aligned}\int_0^C dCf &= -\frac{C}{G_{Bab}G_{Bcd}} + \frac{4}{C}\ln\Big(1 - \frac{C^2}{4G_{Bab}G_{Bcd}}\Big)\\ &\quad + \frac{4}{\sqrt{G_{Bab}G_{Bcd}}}\,\mathrm{arctanh}\Big(\frac{1}{2}\frac{C}{\sqrt{G_{Bab}G_{Bcd}}}\Big)\end{aligned} \tag{8.42}$$

Inserted in the second equation of (8.41) this leaves us with three two–parameter integrals, of which the first one is elementary. Applying the substitution

$$y = \frac{c(1-a)}{a(1-c)} \tag{8.43}$$

to the second integral, and

$$y^2 = \frac{c(1-a)}{a(1-c)} \tag{8.44}$$

to the third integral, those are transformed into known standard integrals, tabulated for instance in [240]. The result is

$$\int_{NP} f = 12\zeta(3) - 8\zeta(2) \tag{8.45}$$

The calculation in the planar sector is elementary, and we just give the result,

$$\int_P f = 4\zeta(2) - 4 \tag{8.46}$$

Putting the pieces together, we have, up to terms of order $O(\epsilon^0)$,

$$\Gamma^{(3)}_{\rm vac}(D) = m^{3D-12}(4\pi)^{-\frac{3}{2}D}\Gamma(6 - \frac{3}{2}D)\Big\{\Big[\frac{2B(2-\frac{D}{2}, 2-\frac{D}{2})}{D-2}\Big]^2 + 12\zeta(3) - 4\zeta(2) - 4\Big\} \tag{8.47}$$

This calculation method generalizes in an obvious way to the tensor integrals which appear in worldline calculations of three-loop renormalization group functions in other abelian theories such as QED or the Yukawa model. The basic scalar integral considered here appears in the calculation of the 3 - loop β – function for ϕ^4 - theory. This permits an easy check of the above calculation against a Feynman diagram calculation. In diagrammatic terms, the integral $\Gamma^{(3)}_{\rm vac}(D)$ corresponds to a weighted sum of the two scalar 3-loop vertex diagrams depicted in fig. 24, calculated at zero external momentum, with massive propagators along the loop, and massless propagator insertions.

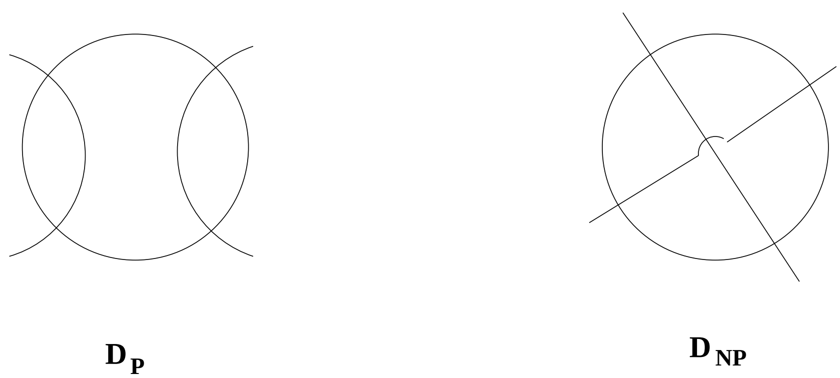


Figure 24: 3-loop ϕ^4 vertex diagrams.

It is not difficult to verify that, with an appropriate normalization, the relation

$$\Gamma^{(3)}_{\rm vac}(D) = 16 D_P + 8 D_{NP} \tag{8.48}$$

indeed holds true for the singular parts of the $\frac{1}{\epsilon}$ – expansions.

9. The QED Photon S-Matrix

We generalize this "multiloop worldline formalism" to the case of photon scattering in quantum electrodynamics [85,92].

9.1. The Single Scalar Loop

We begin with studying scalar electrodynamics at the two-loop level, i.e. a scalar loop with an internal photon correction. A photon insertion in the worldloop may, in Feynman gauge, be represented in terms of the following current-current interaction term inserted into the one-loop path integral,

$$-\frac{e^2}{2}\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}}\int_0^T d\tau_a \int_0^T d\tau_b \frac{\dot{x}(\tau_a)\cdot\dot{x}(\tau_b)}{\Bigl([x(\tau_a)-x(\tau_b)]^2\Bigr)^{\lambda}} \tag{9.1}$$

($\lambda = \frac{D}{2}-1$). This is essentially still Feynman's formula eq.(1.7), except that we have rewritten it in D dimensions, and Euclidean conventions.

As in the case of the scalar propagator, we can exponentiate the offending "non-Gaussian" denominator,

$$\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}\Bigl([x(\tau_a)-x(\tau_b)]^2\Bigr)^{\lambda}} = \int_0^\infty d\bar{T}(4\pi\bar{T})^{-\frac{D}{2}}\exp\Biggl[-\frac{\Bigl(x(\tau_a)-x(\tau_b)\Bigr)^2}{4\bar{T}}\Biggr] \tag{9.2}$$

and absorb it into the worldline Green's function. We obtain then, of course, the same 2-loop worldline Green's function eq.(8.10) and determinant factor eq.(8.11) as before. The numerator $\dot{x}_a\cdot\dot{x}_b$ remains, and will participate in the Wick contractions.

This treatment of the photon propagator may appear somewhat unnatural, but will be seen to work quite well in practice. Moreover, only this procedure will enable us to use the same universal Green's functions both for scalar field theory and gauge theory calculations.

As in the scalar field theory case, the generalization from one-loop to two-loop calculations of photon amplitudes in scalar QED requires no changes of the formalism itself, but only of the Green's functions used, and of the global determinant factor.

The generalization to an arbitrary fixed number of photon insertions is obvious. To obtain a parameter integral representation for the sum of all diagrams with one scalar loop and fixed numbers of photons, N external and m internal, we have to Wick contract N photon vertex operators, together with m factors of $\int_0^T d\tau_a \int_0^T d\tau_b\, \dot{x}_a\cdot\dot{x}_b$, using the $(m+1)-$ loop Green's function $G^{(m)}$:

$$\begin{aligned}\Gamma^{(m+1)}_{\rm scal}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \;&=\; (-ie)^N\Bigl(-\frac{e^2}{2}\Bigr)^m\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}T^{-\frac{D}{2}}(4\pi)^{-(m+1)\frac{D}{2}}\prod_{j=1}^m\int_0^\infty d\bar{T}_j\\ &\quad\times\int_0^T d\tau_{a_j}\int_0^T d\tau_{b_j} N^{(m)\,\frac{D}{2}}\Bigl\langle\prod_{j=1}^m \dot{x}_{a_j}\cdot\dot{x}_{b_j} V^A_{\rm scal,1}\cdots V^A_{{\rm scal},N}\Bigr\rangle\end{aligned} \tag{9.3}$$

This is our $(m+1)$ – loop generalization of the one – loop photon scattering formula eq.(4.22) (to avoid a further complication of nomenclature, it should simply be understood in the following that this is only the "quenched" part of the amplitude). As in the one-loop case, one could immediately translate this into a master formula of the type eq.(1.18).

It is important to note that precisely the same integral representation could be obtained starting from the one-loop formula eq.(4.22) with $(N+2m)$ external photons, and then sewing together m pairs of them, using Feynman gauge. Of course this would be much more laborious. This also explains why the multiloop Green's functions can be rewritten in terms of the one-loop Green's function G_B. It has the nontrivial consequence that one is still allowed to use the one-loop replacement rule; after writing out the result of the Wick – contractions in terms of the one-loop Green's function G_{Bij}, one can generate the corresponding spinor loop integrand by the usual partial integration routine, and use of eq.(2.15).

While this multiloop construction is done most simply using Feynman gauge for the propagator insertions, other gauges can be implemented as well (the gauge freedom was also discussed in [91]). In an arbitrary covariant gauge, the photon insertion term eq.(9.1) would read

$$
\begin{aligned}
-\frac{e^2}{2}\frac{1}{4\pi^{\frac{D}{2}}}\int_0^T d\tau_a\int_0^T d\tau_b\biggl\{&\frac{1+\alpha}{2}\Gamma\Bigl(\frac{D}{2}-1\Bigr)\frac{\dot{x}_a\cdot\dot{x}_b}{\Bigl[(x_a-x_b)^2\Bigr]^{\frac{D}{2}-1}}\\
&+(1-\alpha)\Gamma\Bigl(\frac{D}{2}\Bigr)\frac{\dot{x}_a\cdot(x_a-x_b)(x_a-x_b)\cdot\dot{x}_b}{\Bigl[(x_a-x_b)^2\Bigr]^{\frac{D}{2}}}\biggr\}
\end{aligned}
\tag{9.4}
$$

Here $\alpha=1$ corresponds to Feynman gauge, $\alpha=0$ to Landau gauge. The integrand may also be written as

$$
\Gamma\Bigl(\frac{D}{2}-1\Bigr)\frac{\dot{x}_a\cdot\dot{x}_b}{\Bigl[(x_a-x_b)^2\Bigr]^{\frac{D}{2}-1}}\quad-\frac{1-\alpha}{4}\Gamma\Bigl(\frac{D}{2}-2\Bigr)\frac{\partial}{\partial\tau_a}\frac{\partial}{\partial\tau_b}\Bigl[(x_a-x_b)^2\Bigr]^{2-\frac{D}{2}}
\tag{9.5}
$$

This shows that, on the worldline, gauge transformations correspond to the addition of total derivative terms. This form of the photon insertion is also the more practical one for actual calculations. The power of $(x_a-x_b)^2$ appearing in the second term is then again to be exponentiated.

Before leaving this section, let us mention that sometimes it can be useful to exponentiate also the numerator of the inserted Feynman propagator, rewriting

$$
\dot{x}_a\cdot\dot{x}_b=\frac{1}{2}\lim_{\tau_a'\to\tau_a}\lim_{\tau_b'\to\tau_b}\lim_{\alpha\to 0}\frac{\partial}{\partial\alpha}\frac{\partial}{\partial\tau_a'}\frac{\partial}{\partial\tau_b'}\,\mathrm{e}^{-\alpha(x_{a'}-x_{b'})^2}
\tag{9.6}
$$

This little point-splitting trick turns out to be surprisingly useful for the computerization of the algorithm [241]. For example, it allows one to generate the integrand for the 3-loop scalar QED vacuum amplitude by differentiations performed on the 5-loop determinant factor $N^{(4)}$, instead of Wick – contractions at the 3 – loop level.

9.2. The Single Electron Loop

As in the one-loop case, the transition to spinor electrodynamics is most simply accomplished by supersymmetrization. According to the supersymmetrization rules, the photon insertion eq.(9.1) generalizes to the spinor loop as follows,

$$\frac{e^2}{2}\frac{\Gamma(\lambda)}{4\pi^{\lambda+1}}\int_0^T d\tau_a\,d\theta_a\int_0^T d\tau_b\,d\theta_b\frac{DX_a\cdot DX_b}{\bigl((X_a-X_b)^2\bigr)^{\lambda}} \tag{9.7}$$

The simplest way to verify the correctness of this expression is to write the one-loop two-photon amplitude in the super-formalism, and then sewing together the external legs to create an internal photon, using Feynman gauge.

Just as a demonstration of the usefulness of the superfield formalism, let us rewrite the double integral in components:

$$\begin{aligned}\int_0^T d\tau_a\int_0^T d\tau_b\Biggl\{&-\frac{\dot x_a^\mu \dot x_{b\mu}}{\bigl((x_a-x_b)^2\bigr)^{\lambda}}-4\lambda\frac{(x_a^\mu-x_b^\mu)(\psi_b^\mu\psi_b^\nu\dot x_{a\nu}-\psi_a^\mu\psi_a^\nu\dot x_{b\nu})}{\bigl((x_a-x_b)^2\bigr)^{\lambda+1}}\\ &+8\lambda\frac{(\psi_a^\mu\psi_{b\mu})^2}{\bigl((x_a-x_b)^2\bigr)^{\lambda+1}}-16\lambda(\lambda+1)\frac{(x_a^\mu-x_b^\mu)(x_a^\nu-x_b^\nu)\psi_{a\mu}\psi_{b\nu}\psi_a^\kappa\psi_{b\kappa}}{\bigl((x_a-x_b)^2\bigr)^{\lambda+2}}\Biggr\}\end{aligned} \tag{9.8}$$

The denominator of eq.(9.7) being bosonic, we can again use the proper-time representation eq.(9.2) to get it into the exponent, and then absorb this exponent into the worldline superpropagator. The algebra is completely identical to the scalar case, and leads to modified superpropagators $\hat{G}^{(m)}$ which are given by the same formulas as in eqs.(8.10) and (8.26), with all the one-loop Green's functions appearing on the right-hand sides replaced by the corresponding one-loop superpropagators eq.(4.31). The same applies to the determinant factor $(N^{(m)})^{\frac{D}{2}}$. The generalization of the $(m+1)$ – loop N – photon scattering formula eq.(9.3) to the spinor loop case is equally trivial, and there is no point in writing it down here.

Again what we have at hand is a parameter integral combining into one formula all Feynman diagrams with one electron loop, and fixed numbers of external and internal photons. For instance, for $N = m = 2$ this just corresponds to the diagrams of fig. 8 which we discussed in the introduction.

Finally, note that the formulas (9.4),(9.5) for an arbitrary covariant gauge also carry over to the spinor loop case *mutatis mutandis*.

9.3. The General Case

The general case of a multiple product of scalar or electron loop path integrals coupled by photon insertions requires only two new considerations.

Firstly, every scalar/electron path integral has its own zero-mode integral, which must be separated off, and yields momentum conservation for the photon momenta entering that particular loop. Total momentum conservation is obtained only after all zero mode integrals are performed.

Secondly, since we now have to Wick – contract vertex operators attached to the different loops, the multiloop worldline Green's function becomes a matrix in the space of loops. For instance, in the case of just two scalar/electron loops one has a two by two matrix of Green's functions $G_B^{\alpha\beta}$. The matrix element G_B^{11} has to be used for the Wick contraction of two photon vertex operators both on loop 1, G_B^{12} for the contraction of one vertex operator on loop 1 and one on loop 2, etc. The explicit expressions for these Green's functions can be found by the same procedure which we described in the previous chapter. This leads to formulas similar to eqs. (8.26),(8.27), which we will not write down here (the simplest case of two loops connected by a single propagator was also considered in [91]).

Note that in the QED case we encounter neither of the two problems discussed in the last chapter, which motivated the introduction of an auxiliary field formalism. First, Feynman's formula eq.(1.7) and its supersymmetrization allow one to neatly exhaust the complete photon S-matrix in terms of scalar/electron path integrals connected by photon insertions. The question of non-Hamiltonian graphs therefore does not arise. Moreover, from the same formula it follows that the contributions of individual Feynman diagrams are always generated with the appropriate statistical factors.

Note that we do not consider here amplitudes involving external electrons; the corresponding formalism has not yet been sufficiently developed. For a treatment of external scalars in scalar QED along the present lines see [91].

9.4. Example: The 2-Loop QED β – Functions

As an illustration, we will use this calculus for a re-derivation of the two-loop QED β-function, both for scalar and for spinor electrodynamics [85,93].

As usual, matters much simplify if one is interested only in the β-function contribution, as opposed to a calculation of the complete 2-loop vacuum polarization amplitude. Our strategy here will be to use the effective action formalism with a constant background field $F_{\mu\nu}$, and read off the β-function from the coefficient of the induced $F_{\mu\nu}F^{\mu\nu}$-term. Standard dimensional regularization will be used for the treatment of the UV divergences.

Just for setting the stage, let us first redo the one-loop calculation. As always in the constant field case we choose Fock-Schwinger gauge centered at x_0, so that $A_\mu = \frac{1}{2}y^\rho F_{\rho\mu}$. Using this A – field in the spinor-loop path-integral eq.(1.9), expanding the interaction exponential to second order, and performing the Wick contractions, one obtains

$$
\begin{aligned}
\Gamma^{(1)}_{\rm spin}[F] &= -\frac{1}{2}\int_0^\infty \frac{dT}{T} e^{-m^2T}\int \mathcal{D}x\mathcal{D}\psi \exp\Big[-\int_0^T d\tau\Big(\frac{1}{4}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi}\Big)\Big] \\
&\quad \times(-\frac{e^2}{2})\int_0^T d\tau_1\int_0^T d\tau_2\Big[\frac{1}{4}\dot{x}_1^\mu F_{\mu\nu}x_1^\nu \dot{x}_2^\alpha F_{\alpha\beta}x_2^\beta + \psi_1^\mu F_{\mu\nu}\psi_1^\nu\psi_2^\alpha F_{\alpha\beta}\psi_2^\beta\Big] \\
&= \frac{e^2}{2}\int_0^\infty \frac{dT}{T}(4\pi T)^{-\frac{D}{2}} e^{-m^2T}\int_0^T d\tau_1\int_0^T d\tau_2\Big(\dot{G}_{B12}^{\,2} - G_{F12}^2\Big)\int dx_0 F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{9.9}
$$

The parameter integral gives

$$\int_0^T d\tau_1 \int_0^T d\tau_2 \Bigl(\dot{G}_B^2(\tau_1,\tau_2) - G_F^2(\tau_1,\tau_2)\Bigr) = -\frac{2}{3}T^2 \tag{9.10}$$

The singular part of the one-loop effective action becomes

$$\Gamma^{(1)}_{\rm spin}[F] \sim \frac{2}{3}(4\pi)^{-2}\frac{1}{\epsilon}e^2 \int dx_0 F_{\mu\nu}F^{\mu\nu} \tag{9.11}$$

($\epsilon = D-4$). From this one can read off the one-loop photon wave-function renormalization factor

$$(Z_3-1)^{(1)} = \frac{2}{3\epsilon}\frac{\alpha}{\pi} \tag{9.12}$$

leading to the usual value for the one-loop spinor QED β – function,

$$\beta^{(1)}_{\rm spin}(\alpha) = \frac{2}{3}\frac{\alpha^2}{\pi} \tag{9.13}$$

($\alpha = \frac{e^2}{4\pi}$).

The corresponding result for scalar QED is simply obtained by omitting, in eq.(9.9), the term involving G_F, and the global factor of -2. This yields

$$\beta^{(1)}_{\rm scal}(\alpha) = \frac{1}{6}\frac{\alpha^2}{\pi} \tag{9.14}$$

Now let us tackle the two-loop calculation. In the corresponding Feynman diagram calculation (see, e.g., [117]), one would have to separately calculate the three diagrams of fig. 25, and then extract their $\frac{1}{\epsilon}$ – poles. Cancellation of the $\frac{1}{\epsilon^2}$ – poles would be found in the sum of the results, indicating a cancellation of subdivergences due to gauge invariance.

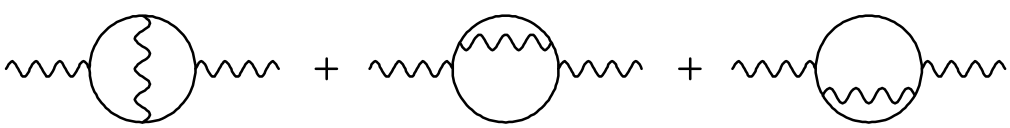


Figure 25: Diagrams contributing to the two-loop vacuum polarization.

Let us begin with the purely bosonic contributions, which correspond to the scalar QED calculation. Those are obtained by inserting the worldline current-current interaction term eq.(9.1) into the bosonic one-loop path-integral. After exponentiation of the denominator and absorption into the worldline Green's function, this results in

$$
\begin{aligned}
\Gamma^{(2)}_{\rm bos}[F] &= -2\Gamma^{(2)}_{\rm scal}[F] \\
&= -2(4\pi)^{-D}\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar T\int_0^T d\tau_a\int_0^T d\tau_b\,[\bar T+G_{Bab}]^{-\frac{D}{2}} \\
&\quad\times(\frac{-e^2}{2})^2\int_0^T d\tau_1\int_0^T d\tau_2\int dx_0\frac{1}{4}\langle\dot y_1^\mu F_{\mu\nu}y_1^\nu\dot y_2^\alpha F_{\alpha\beta}y_2^\beta\dot y_a^\lambda\dot y_{b\lambda}\rangle
\end{aligned}
\tag{9.15}
$$

Note the appearance of the two-loop determinant factor $[\bar T+G_B(\tau_a,\tau_b)]^{-\frac{D}{2}}$. The Wick contraction of

$$
\langle\dot y_1^\mu y_1^\nu\dot y_2^\alpha y_2^\beta\dot y_a^\lambda\dot y_{b\lambda}\rangle
\tag{9.16}
$$

has now to be done, using the two-loop Green's function eq.(8.10). Due to the symmetries of the problem there are only two nonequivalent contraction possibilities, namely

$$
\begin{aligned}
\langle\dot y_1^\mu y_2^\beta\rangle\langle y_1^\nu\dot y_2^\alpha\rangle\langle\dot y_a^\lambda\dot y_{b\lambda}\rangle &= -Dg^{\mu\beta}g^{\nu\alpha}\partial_1G^{(1)}_{B12}\partial_2G^{(1)}_{B12}\partial_a\partial_bG^{(1)}_{Bab} \\
\langle\dot y_1^\mu\dot y_2^\alpha\rangle\langle y_1^\nu\dot y_a^\lambda\rangle\langle y_2^\beta\dot y_{b\lambda}\rangle &= -g^{\mu\alpha}g^{\nu\beta}\partial_1\partial_2G^{(1)}_{B12}\partial_aG^{(1)}_{B1a}\partial_bG^{(1)}_{B2b}
\end{aligned}
\tag{9.17}
$$

Those occur with multiplicities 2 and 8, respectively. Care must be taken with Wick contractions involving $\dot y_a,\dot y_b$, as the derivatives should not act on the τ_a,τ_b explicitly appearing in that Green's function. The result is written out in terms of the bosonic one-loop Green's function and its derivatives. As in the one-loop calculation, one next eliminates all factors of $\ddot G_B$ appearing by partial integrations with respect to $\tau_1,\tau_2,\tau_a,\tau_b$. As the next step, all fermionic contributions are included by applying the one-loop replacement rule (2.15). For example, one replaces

$$
\dot G_{B12}\dot G_{B21}\dot G_{Bab}\dot G_{Bba}\rightarrow(\dot G_{B12}\dot G_{B21}-G_{F12}G_{F21})(\dot G_{Bab}\dot G_{Bba}-G_{Fab}G_{Fba})
\tag{9.18}
$$

etc.

At this stage, we have the desired contribution to the two-loop effective action in the form of a sixfold integral (see fig. 26),

$$
\begin{aligned}
{\cal L}^{(2)}_{\rm spin}[F] &= -2(4\pi)^{-D}\frac{e^4}{16}\int_0^\infty\frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar T \\
&\quad\times\int_0^T d\tau_ad\tau_bd\tau_1d\tau_2\,P(T,\bar T,\tau_a,\tau_b,\tau_1,\tau_2)F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{9.19}
$$

The integrand function P is a polynomial in the various $G_{Bij},\dot G_{Bij},G_{Fij}$, multiplied by powers of $\gamma\equiv\Bigl[\bar T+G_B(\tau_a,\tau_b)\Bigr]^{-1}$. Let us just write down its purely bosonic part $P_{\rm bos}$, which is [31]

[31] In [85] this integrand was given incorrectly.

$$
\begin{aligned}
P_{\rm bos} \;=\; & \gamma^{\frac{D}{2}} \Big\{ D(D-1)\gamma \dot{G}^2_{Bab}\dot{G}^2_{B12} + 8D\gamma \dot{G}_{Bab}\dot{G}_{B12}\dot{G}_{B1a}\dot{G}_{B2b} \\
& +8\gamma \dot{G}_{B1a}\dot{G}_{Bab}\dot{G}_{B12}[\dot{G}_{B2a} - \dot{G}_{B2b}] \\
& -4\gamma \dot{G}_{B1a}\dot{G}_{B2b}[\dot{G}_{B1a} - \dot{G}_{B1b}][\dot{G}_{B2a} - \dot{G}_{B2b}] \\
& +(D+2)(D-1)\gamma^2 \dot{G}^2_{Bab}\dot{G}_{B12}[\dot{G}_{B1a} - \dot{G}_{B1b}][G_{B2a} - G_{B2b}] \Big\}
\end{aligned}
\tag{9.20}
$$

In writing this polynomial, we have used the symmetry with regard to interchange of τ_1 and τ_2 to combine some terms, and omitted some terms which are total derivatives with respect to $\int d\tau_1$ or $\int d\tau_2$ (those terms are easy to identify at an early stage of the calculation).

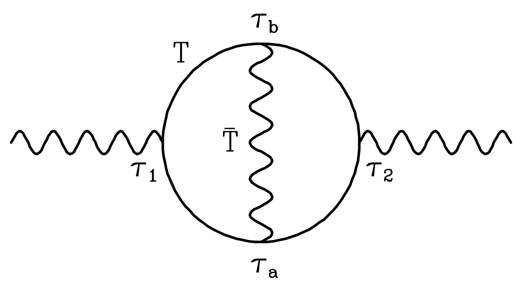


Figure 26: Definition of the six integration parameters.

It is convenient to begin with the integrations over τ_1, τ_2. Those are polynomial, and easily performed using a set of relations of the type

$$
\begin{aligned}
\int_0^1 du_2 \dot{G}_{B12}\dot{G}_{B23} &= 2G_{B13} - \frac{1}{3} \\
\int_0^1 du_2 G_{B12}G_{B23} &= -\frac{1}{6}G^2_{B13} + \frac{1}{30} \\
\vdots \quad & \quad \vdots
\end{aligned}
\tag{9.21}
$$

All those relations may be derived from the following master identities proven in appendix B,

$$
\begin{aligned}
\int_0^1 du_2 \dots du_n \dot{G}_{B12}\dot{G}_{B23}\dots \dot{G}_{Bn(n+1)} &= -\frac{2^n}{n!}B_n(|u_1 - u_{n+1}|)\mathrm{sign}^n(u_1 - u_{n+1}) \\
\int_0^1 du_2 \dots du_n G_{F12}G_{F23}\dots G_{Fn(n+1)} &= \frac{2^{n-1}}{(n-1)!}E_{n-1}(|u_1 - u_{n+1}|)\mathrm{sign}^n(u_1 - u_{n+1})
\end{aligned}
\tag{9.22}
$$

In writing these identities, we have scaled down to the unit circle again. B_n denotes the n^{th} Bernoulli-polynomial, and E_n the n^{th} Euler-polynomial. Due to the fact that those polynomials can be rewritten as

$$
\begin{aligned}
B_n(x) &= P_n(x^2-x) \qquad (\text{n} \quad \text{even}) \\
B_n(x) &= P_n(x^2-x)(x-\frac{1}{2}) \quad (\text{n} \quad \text{odd})
\end{aligned}
\tag{9.23}
$$

with another set of polynomials $P_n(x)$ (the same property holds true for $E_n(x)$), the right hand sides can always be re-expressed in terms of $G_B, \dot{G}_B$ and G_F, so that explicit u_i's will never appear in those relations. Those integrals needed for the present calculation are listed in appendix F.

Next we perform the $\bar{T}$ – integration, which is trivial:

$$
\int_0^\infty d\bar{T}[\bar{T}+G_{Bab}]^{-\frac{D}{2}-k} = \frac{G_{Bab}^{1-\frac{D}{2}-k}}{\frac{D}{2}+k-1} \qquad (k=1,2) \tag{9.24}
$$

Collecting terms, and using (F.25), we get

$$
\begin{aligned}
\int_0^\infty d\bar{T}\int_0^T d\tau_1 \int_0^T d\tau_2\, P(T,\bar{T},\tau_a,\tau_b,\tau_1,\tau_2) \quad &= \\
\frac{16}{3D}\Bigl\{(D-4)(D-1)G_{Bab}^{1-\frac{D}{2}}T \quad &+ \quad (D-2)(D-7)G_{Bab}^{2-\frac{D}{2}}\Bigr\}
\end{aligned}
\tag{9.25}
$$

The corresponding expression for scalar QED is obtained by using only the bosonic part $P_{\rm bos}$ of the function P:

$$
\begin{aligned}
\int_0^\infty d\bar{T}\int_0^T d\tau_1 \int_0^T d\tau_2\, P_{\rm bos}(T,\bar{T},\tau_a,\tau_b,\tau_1,\tau_2) &= \frac{2}{3}(D-1)G_{Bab}^{-\frac{D}{2}}T^2 \\
+(D-1)(\frac{32}{3D}-4)G_{Bab}^{1-\frac{D}{2}}T + \frac{16}{3D}&(D-2)(D-7)G_{Bab}^{2-\frac{D}{2}}
\end{aligned}
\tag{9.26}
$$

Setting $\tau_a = 0$, the integration over τ_b produces a couple of Euler Beta-functions,

$$
\int_0^T d\tau_a \int_0^T d\tau_b\, G_{Bab}^{k-\frac{D}{2}} = B\Bigl(k+1-\frac{D}{2}, k+1-\frac{D}{2}\Bigr)T^{2+k-\frac{D}{2}} \tag{9.27}
$$

As in the one-loop case, the remaining electron proper-time integral just gives a Γ – function:

$$
\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}T^{4-D} = \Gamma(4-D)m^{2(D-4)} \tag{9.28}
$$

Combining terms and performing the ϵ – expansions for the effective Lagrangians, we obtain

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm scal}[F] &\sim \frac{1}{2\epsilon} e^4 (4\pi)^{-4} F_{\mu\nu}F^{\mu\nu} + O(\epsilon^0) \\
\mathcal{L}^{(2)}_{\rm spin}[F] &\sim -\frac{3}{\epsilon} e^4 (4\pi)^{-4} F_{\mu\nu}F^{\mu\nu} + O(\epsilon^0)
\end{aligned}
\tag{9.29}
$$

So far this is a calculation of the bare regularized effective action. What about renormalization? The counterdiagrams due to electron wave function and vertex renormalization need not be taken into account, since they cancel by the QED Ward identity ($Z_1 = Z_2$). However, we have used the electron mass as an infrared regulator for the electron proper-time integral eq.(9.28); mass renormalization must therefore be dealt with.

Since our calculation corresponds to a field theory calculation in dimensional regularization, we need to know the corresponding one-loop mass renormalization counterterms, both for scalar and spinor QED. This is a simple textbook calculation, of which we give the result only:

$$
\begin{aligned}
\frac{\delta m^2_{\rm scal}}{m^2_{\rm scal}} &= \frac{6}{\epsilon} e^2 (4\pi)^{-2} \\
\frac{\delta m_{\rm spin}}{m_{\rm spin}} &= \frac{6}{\epsilon} e^2 (4\pi)^{-2}
\end{aligned}
\tag{9.30}
$$

We insert those counterterms into the one-loop path integrals, and obtain the following additional contributions to the two-loop effective Lagrangians,

$$
\begin{aligned}
\Delta\Gamma^{(2)}_{\rm scal}[F] &= \delta m^2_{\rm scal} \frac{\partial}{\partial m^2} \Gamma^{(1)}_{\rm scal}[F] \\
&\sim \frac{1}{2\epsilon} e^4 (4\pi)^{-4} \int dx_0 F_{\mu\nu}F^{\mu\nu} + O(\epsilon^0) \\
\Delta\Gamma^{(2)}_{\rm spin}[F] &= \delta m_{\rm spin} \frac{\partial}{\partial m} \Gamma^{(1)}_{\rm spin}[F] \\
&\sim \frac{4}{\epsilon} e^4 (4\pi)^{-4} \int dx_0 F_{\mu\nu}F^{\mu\nu} + O(\epsilon^0)
\end{aligned}
\tag{9.31}
$$

The extraction of the β − function coefficients proceeds in the usual way. From the total effective Lagrangians

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm scal}[F] + \Delta\mathcal{L}^{(2)}_{\rm scal}[F] &\sim \frac{1}{\epsilon} e^4 (4\pi)^{-4} F_{\mu\nu}F^{\mu\nu} \\
\mathcal{L}^{(2)}_{\rm spin}[F] + \Delta\mathcal{L}^{(2)}_{\rm spin}[F] &\sim \frac{1}{\epsilon} e^4 (4\pi)^{-4} F_{\mu\nu}F^{\mu\nu}
\end{aligned}
\tag{9.32}
$$

one obtains the two-loop photon wave-function renormalization factors, and from those the standard results for the two-loop β – function coefficients [242,243],

$$\beta^{(2)}_{\rm scal}(\alpha) = \beta^{(2)}_{\rm spin}(\alpha) = \frac{\alpha^3}{2\pi^2} \tag{9.33}$$

Observe that in the spinor-loop case, the integrand after performance of the first three integrations, eq.(9.25), has only one term which would be divergent for $D = 4$ when integrated over τ_b. Moreover, the coefficient of this term vanishes for $D = 4$. This suggests that this calculation can be further simplified by using some four-dimensional regularization scheme. And indeed, if we do the spinor-loop calculation in four dimension, then instead of eq.(9.25) we find simply

$$\int_0^\infty d\bar{T} \int_0^T d\tau_1 \int_0^T d\tau_2 \, P(T, \bar{T}, \tau_a, \tau_b, \tau_1, \tau_2) = -8 \tag{9.34}$$

This time there is no dependence on τ_a, τ_b left, so that one immediately gets

$$\mathcal{L}'^{(2)}_{\rm spin}[F] = (4\pi)^{-4} e^4 \int_0^\infty \frac{dT}{T} e^{-m^2 T} F_{\mu\nu} F^{\mu\nu} \tag{9.35}$$

It is only the final electron proper-time integral that now needs to be regularized. This can be done by introducing a proper-time cutoff T_0 at the lower integration limit, which replaces eq.(9.28) by

$$\int_{T_0}^\infty \frac{dT}{T} {\rm e}^{-m^2 T} \sim -\ln(m^2 T_0) \tag{9.36}$$

(Pauli-Villars regularization could be used as well, although proper-time regularization appears more natural in the worldline formalism). With this regulator, the two-loop effective Lagrangian becomes

$$\mathcal{L}'^{(2)}_{\rm spin}[F] \sim -\ln(m^2 T_0)(4\pi)^{-4} e^4 F_{\mu\nu} F^{\mu\nu} + \text{finite} \tag{9.37}$$

In spite of the manifest suppression of subdivergences, there is again a contribution from mass renormalization, which can be determined by comparison with the corresponding Feynman calculation. On-shell renormalization of spinor QED using a proper-time cutoff has been studied in [158,172]. It leads to a one-loop mass renormalization counterterm

$$\frac{\delta m}{m} = 3\ln(m^2 T_0) e^2 (4\pi)^{-2} + \text{finite} \tag{9.38}$$

Insertion of this counterterm into the one-loop path integral gives

$$\begin{aligned} \Delta\Gamma'^{(2)}_{\rm spin}[F] &= \delta m \frac{\partial}{\partial m} \Gamma'^{(1)}[F] \\ &\sim 2\ln(m^2 T_0)(4\pi)^{-4} e^4 \int dx_0 F_{\mu\nu} F^{\mu\nu} + \text{finite} \end{aligned} \tag{9.39}$$

so that mass renormalization now just amounts to a sign change for the effective Lagrangian:

$$\mathcal{L}'^{(2)}_{\rm spin}[F] + \Delta\mathcal{L}'^{(2)}_{\rm spin}[F] \sim \ln(m^2 T_0)(4\pi)^{-4} e^4 F_{\mu\nu}F^{\mu\nu} \tag{9.40}$$

The extraction of the (still scheme-independent) β – function coefficient $\beta^{(2)}_{\rm spin}(\alpha)$ is again standard [172], and leads back to eq.(9.33).

Let us summarize the properties of this calculation:

1. Neither momentum integrals nor Dirac traces had to be calculated.
2. The three diagrams of fig. 25 were combined into one calculation (in fact, in this formalism it is somewhat *easier* to compute the sum than any single one of them).
3. In the spinor-loop case, we have managed to obtain the correct 2-loop coefficient without performing any nontrivial integrals.

We will see later on that the cancellations which led to the last property are not accidental, but a direct consequence of renormalizability.

9.5. Quantum Electrodynamics in a Constant External Field

As in the one-loop case, it takes only minor modifications to extend this formalism to the case when an additional constant external field $F_{\mu\nu}$ is present [92].

Again we begin with the simplest case, which is scalar QED at the 2-loop level. Combining the contributions to the quadratic part of the worldline Lagrangian due to the external field eq.(5.2) and to the propagator insertion eq. (9.2), we obtain the new total bosonic kinetic operator

$$\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau} - \frac{B_{ab}}{\bar{T}} \tag{9.41}$$

The inverse of this operator can still be constructed as a geometric series,

$$\begin{aligned}\Bigl(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau} - \frac{B_{ab}}{\bar{T}}\Bigr)^{-1} \quad &= \quad \Bigl(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\Bigr)^{-1} \\ &+ \Bigl(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\Bigr)^{-1}\frac{B_{ab}}{\bar{T}}\Bigl(\frac{d^2}{d\tau^2} - 2ieF\frac{d}{d\tau}\Bigr)^{-1} + \cdots\end{aligned} \tag{9.42}$$

Summing this series one obtains the following Green's function:

$$\mathcal{G}^{(1)}_B(\tau_1,\tau_2) = \mathcal{G}_B(\tau_1,\tau_2) + \frac{1}{2}\frac{[\mathcal{G}_B(\tau_1,\tau_a) - \mathcal{G}_B(\tau_1,\tau_b)][\mathcal{G}_B(\tau_a,\tau_2) - \mathcal{G}_B(\tau_b,\tau_2)]}{\bar{T} - \frac{1}{2}\mathcal{C}_{ab}} \tag{9.43}$$

were we have defined

$$\begin{aligned}\mathcal{C}_{ab} &\equiv \mathcal{G}_B(\tau_a,\tau_a)-\mathcal{G}_B(\tau_a,\tau_b)-\mathcal{G}_B(\tau_b,\tau_a)+\mathcal{G}_B(\tau_b,\tau_b)\\ &= T\frac{\cos(\mathcal{Z})-\cos(\mathcal{Z}\dot{G}_{Bab})}{(\mathcal{Z})\sin(\mathcal{Z})}\end{aligned} \tag{9.44}$$

This is almost but not quite identical with what one would obtain from the ordinary bosonic two-loop Green's function, eq.(8.10), by simply replacing all G_{Bij}'s appearing there by the corresponding $\mathcal{G}_{Bij}$'s. The more complicated structure of the denominator is due to the fact that the $\mathcal{G}_{Bij}$'s are not any more symmetric under the interchange $i \leftrightarrow j$, rather we have $\mathcal{G}_{Bij} = \mathcal{G}^T_{Bji}$, and moreover have non-vanishing coincidence limits. The denominator is now in general a nontrivial Lorentz matrix, and must be interpreted as a matrix inverse (of course, all matrices appearing here commute with each other).

The free Gaussian path integral is again easily calculated using the $ln\, det = tr\, ln$ – identity, yielding

$$\begin{aligned}\mathrm{Det}'_P\Bigl[-\frac{d^2}{d\tau^2}+2ieF\frac{d}{d\tau}+\frac{B_{ab}}{\bar{T}}\Bigr] &= \mathrm{Det}'_P\Bigl[-\frac{d^2}{d\tau^2}\Bigr]\mathrm{Det}'_P\Bigl[\mathbb{1}-2ieF\Bigl(\frac{d}{d\tau}\Bigr)^{-1}\Bigr]\\ &\quad\times\mathrm{Det}'_P\Bigl[\mathbb{1}-\frac{B_{ab}}{\bar{T}}\Bigl(\frac{d^2}{d\tau^2}-2ieF\frac{d}{d\tau}\Bigr)^{-1}\Bigr]\\ &= (4\pi T)^D\det\Bigl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Bigr]\det\Bigl[\mathbb{1}-\frac{1}{2\bar{T}}\mathcal{C}_{ab}\Bigr]\end{aligned} \tag{9.45}$$

Thus we have now a product of two Lorentz matrix determinants. The first one is identical with the by now familiar Euler-Heisenberg integrand, eq.(5.16), while the second one generalizes the two-loop determinant factor eq.(8.11) to the external field case.

As in the case without a background field, the whole procedure goes through essentially unchanged for the fermion loop, if the superfield formalism is used. As a consequence, one finds the same close relationship as before between the parameter integrals for the same amplitude calculated for the scalar and for the fermion loop: They differ only by a replacement of all $\mathcal{G}_B$'s by $\hat{\mathcal{G}}$'s, and by the additional θ – integrations. Of course, one must also replace the scalar QED one-loop Euler-Heisenberg factor eq.(5.16) by its spinor QED equivalent eq.(5.17), and as always the global factor of -2 must be taken into account.

The generalization to an arbitrary fixed number of photon insertions is straightforward, and leads to formulas for the generalized (super-) Green's functions and (super-) determinants identical with the ones given above for the vacuum case, up to a replacement of all G_B's ($\hat{G}$'s) by $\mathcal{G}_B$'s ($\hat{\mathcal{G}}$'s). The only point to be mentioned here is that care must now be taken in writing the indices of the $\mathcal{G}_{Bij}$'s appearing. For instance, the (un-subtracted) bosonic three-loop Green's function, eq.(8.26) with $m = 2$, must be written as

$$\mathcal{G}_B^{(2)}(\tau_1,\tau_2) = \mathcal{G}_B(\tau_1,\tau_2)+\frac{1}{2}\sum_{k,l=1}^{2}\Bigl[\mathcal{G}_B(\tau_1,\tau_{a_k})-\mathcal{G}_B(\tau_1,\tau_{b_k})\Bigr]A^{(2)}_{kl}\Bigl[\mathcal{G}_B(\tau_{a_l},\tau_2)-\mathcal{G}_B(\tau_{b_l},\tau_2)\Bigr] \tag{9.46}$$

The matrix A appearing here is the inverse of the matrix

$$\begin{pmatrix} T_1 - \frac{1}{2}\Big(\mathcal{G}_{Ba_1a_1} - \mathcal{G}_{Ba_1b_1} - \mathcal{G}_{Bb_1a_1} + \mathcal{G}_{Bb_1b_1}\Big) & -\frac{1}{2}\Big(\mathcal{G}_{Ba_1a_2} - \mathcal{G}_{Ba_1b_2} - \mathcal{G}_{Bb_1a_2} + \mathcal{G}_{Bb_1b_2}\Big) \\ -\frac{1}{2}\Big(\mathcal{G}_{Ba_2a_1} - \mathcal{G}_{Ba_2b_1} - \mathcal{G}_{Bb_2a_1} + \mathcal{G}_{Bb_2b_1}\Big) & T_2 - \frac{1}{2}\Big(\mathcal{G}_{Ba_2a_2} - \mathcal{G}_{Ba_2b_2} - \mathcal{G}_{Bb_2a_2} + \mathcal{G}_{Bb_2b_2}\Big) \end{pmatrix} \tag{9.47}$$

and T_1, T_2 denote the proper-time lengths of the two inserted propagators.

9.6. Example: The 2-Loop Euler-Heisenberg Lagrangians

As an application of the constant field formalism at the two-loop level, in this section we will calculate the first radiative corrections to the Euler-Heisenberg-Schwinger formulas eqs.(5.23),(5.24) [92,108].

9.6.1. Scalar QED

According to the above, for the scalar QED case we can write this effective Lagrangian in the form

$$\begin{aligned} \mathcal{L}^{(2)}_{\rm scal}[F] &= (4\pi)^{-D}\Big(-\frac{e^2}{2}\Big)\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar{T}\int_0^T d\tau_a\int_0^T d\tau_b \\ &\quad\times{\det}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\Big]{\det}^{-\frac{1}{2}}\Big[\bar{T}-\frac{1}{2}\mathcal{C}_{ab}\Big]\langle\dot{y}_a\cdot\dot{y}_b\rangle \end{aligned} \tag{9.48}$$

In this fourfold parameter integral, T and $\bar{T}$ represent the scalar and photon proper-times, and $\tau_{a,b}$ the endpoints of the photon insertion moving around the scalar loop.

A single Wick contraction is to be performed on the "left-over" numerator of the photon insertion, using the modified worldline Green's function eq.(9.43). This yields

$$\langle\dot{y}_a\cdot\dot{y}_b\rangle = {\rm tr}\Big[\ddot{\mathcal{G}}_{Bab} + \frac{1}{2}\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb})}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Big] \tag{9.49}$$

After replacing the one-loop Green's functions $\dot{\mathcal{G}}_{Bij}$'s as well as $\mathcal{C}_{ab}$ by the explicit expressions given in eqs.(5.9) and eq.(9.44), we have already a parameter integral representation for the bare dimensionally regularized effective Lagrangian.

Alternatively one may remove $\ddot{\mathcal{G}}_B$ by a partial integration with respect to τ_a or τ_b. Using the formula

$$d\det(M) = \det(M){\rm tr}(dMM^{-1}) \tag{9.50}$$

and $\dot{\mathcal{G}}_{Bab} = -\dot{\mathcal{G}}^T_{Bba}$, one obtains the equivalent parameter integral

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm scal}[F] &= (4\pi)^{-D}\Bigl(-\frac{e^2}{2}\Bigr)\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar T\int_0^T d\tau_a\int_0^T d\tau_b\\
&\times{\rm det}^{-\frac{1}{2}}\biggl[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}\biggr]{\rm det}^{-\frac{1}{2}}\biggl[\bar T-\frac{1}{2}\mathcal{C}_{ab}\biggr]\\
&\times\frac{1}{2}\biggl\{{\rm tr}\dot{\mathcal{G}}_{Bab}{\rm tr}\biggl[\frac{\dot{\mathcal{G}}_{Bab}}{\bar T-\frac{1}{2}\mathcal{C}_{ab}}\biggr]+{\rm tr}\biggl[\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb})}{\bar T-\frac{1}{2}\mathcal{C}_{ab}}\biggr]\biggr\}
\end{aligned}
\tag{9.51}
$$

To facilitate the further evaluation and renormalization of this Lagrangian, we specialize the constant field F to a pure magnetic field. It will be instructive to do this calculation in two different regularizations, proper-time and dimensional regularization. The renormalization will be performed on-shell in both cases.

Let us begin with the proper-time regularized version. This regularization keeps the integrations fairly simple, and was used in all previous calculations of two-loop Euler-Heisenberg Lagrangians [244,245,246,247,172]. It means that in the following we set $D=4$, and instead introduce proper-time UV cutoffs for the various proper-time integrals later on.

As in the photon-splitting calculation of section 5.6, we choose the field in the $z-$ direction. For this case the generalized worldline Green's functions and determinants were given in (5.30),(5.31). The combination C_{ab} becomes

$$
\mathcal{C}_{ab} = -2G_{Bab}g_{\parallel} - 2G^z_{Bab}g_{\perp} \tag{9.52}
$$

where

$$
G^z_{Bab} \equiv \frac{T}{2}\frac{\Bigl(\cosh(z)-\cosh(z\dot G_{ab})\Bigr)}{z\sinh(z)} = G_{Bab}-\frac{1}{3T}G^2_{Bab}z^2+O(z^4) \tag{9.53}
$$

We will also use the derivative of this expression,

$$
\dot G^z_{Bab} = \frac{\sinh(z\dot G_{Bab})}{\sinh(z)} \tag{9.54}
$$

Introducing the further abbreviations

$$
\begin{aligned}
\gamma &\equiv (\bar T + G_{Bab})^{-1}\\
\gamma^z &\equiv (\bar T + G^z_{Bab})^{-1}
\end{aligned}
$$

we can then rewrite the various Lorentz traces and determinants appearing in eqs. (9.48), (9.51) as

$$
\begin{aligned}
\det{}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}(\bar{T}-\frac{1}{2}\mathcal{C}_{ab})\Big] &= \frac{z}{\sinh(z)}\gamma\gamma^z \\
\mathrm{tr}\Big[\ddot{\mathcal{G}}_{Bab}\Big] &= 8\delta_{ab}-4-4\frac{z\cosh(z\dot{G}_{Bab})}{\sinh(z)} \\
\frac{1}{2}\mathrm{tr}\dot{\mathcal{G}}_{Bab}\mathrm{tr}\Big[\frac{\dot{\mathcal{G}}_{Bab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Big] &= 2\Big[\dot{G}_{Bab}+\frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)}\Big]\Big[\dot{G}_{Bab}\gamma+\frac{\sinh(z\dot{G}_{Bab})}{\sinh(z)}\gamma^z\Big] \\
\frac{1}{2}\mathrm{tr}\Big[\frac{(\dot{\mathcal{G}}_{aa}-\dot{\mathcal{G}}_{ab})(\dot{\mathcal{G}}_{ab}-\dot{\mathcal{G}}_{bb})}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Big] &= -\gamma^z\frac{\sinh^2(z\dot{G}_{Bab})+\Big[\cosh(z\dot{G}_{Bab})-\cosh(z)\Big]^2}{\sinh^2(z)}-\dot{G}^2_{Bab}\gamma
\end{aligned}
\tag{9.55}
$$

As usual we rescale to the unit circle, $\tau_{a,b}=Tu_{a,b}$, and use translation invariance in τ to set $\tau_b=0$, so that

$$
\begin{aligned}
G_B(\tau_a,\tau_b) &= TG_B(u_a,u_b)=T(u_a-u_a^2) \\
\dot{G}_B(\tau_a,\tau_b) &= \dot{G}_B(u_a,u_b)=1-2u_a
\end{aligned}
$$

After performance of the $\bar{T}$ – integration, which is finite and elementary, eq.(9.51) turns into

$$
\mathcal{L}^{(2)}_{\mathrm{scal}}[B]=-(4\pi)^{-4}\frac{e^2}{2}\int_0^\infty\frac{dT}{T^3}e^{-m^2T}\frac{z}{\sinh(z)}\int_0^1 du_a\,A(z,u_a) \tag{9.56}
$$

with

$$
\begin{aligned}
A &= \Big\{A_1\frac{\ln(G_{Bab}/G^z_{Bab})}{(G_{Bab}-G^z_{Bab})^2}+\frac{A_2}{(G^z_{Bab})(G_{Bab}-G^z_{Bab})}+\frac{A_3}{(G_{Bab})(G_{Bab}-G^z_{Bab})}\Big\} \\
A_1 &= 4\Big[G^z_{Bab}z\coth(z)-G_{Bab}\Big] \\
A_2 &= 1+2\dot{G}_{Bab}\dot{G}^z_{Bab}-4G^z_{Bab}z\coth(z) \\
A_3 &= -\dot{G}^2_{Bab}-2\dot{G}_{Bab}\dot{G}^z_{Bab}
\end{aligned}
\tag{9.57}
$$

All Green's functions now refer to the unit circle, $T=1$. Here and in the following we often use the identity $\dot{G}^2_{Bab}=1-\frac{4}{T}G_{Bab}$ to eliminate $\dot{G}_{Bab}$ in favour of G_{Bab}.

Renormalization must now be addressed, and will be performed in close analogy to the discussion in [172]. The integral in eq.(9.56) suffers from two kinds of divergences:

1. An overall divergence of the loop proper-time integral $\int_0^\infty dT$ at the lower integration limit (which is already familiar from the β – function calculation of section 9.4).
2. Divergences of the $\int_0^1 du_a$ parameter integral at the points $0,1$ where the endpoints of the photon propagator become coincident, $u_a=u_b$.

The first one will be removed by one- and two-loop photon wave function renormalization, the second one by the one-loop renormalization of the scalar mass. As was already mentioned, vertex renormalization and scalar self energy renormalization need not be considered in this type of calculation, since they must cancel on account of the QED Ward identity.

By power counting, an overall divergence can exist only for the terms in the effective Lagrangian which are of order at most quadratic in the external field B. Expanding the integrand of eq.(9.56), $K(z,u_a) \equiv \frac{z}{\sinh(z)}A(z,u_a)$, in the variable z, we find

$$K(z,u_a) = \Bigl[\frac{3}{{G_{Bab}}^2} - \frac{12}{G_{Bab}}\Bigr] + \Bigl[-\frac{1}{2}\frac{1}{G_{Bab}^2} + \frac{1}{G_{Bab}} + 2\Bigr] z^2 + O(z^4) \tag{9.58}$$

The complicated singularity appearing here at the point $u_a = u_b$ indicates that this form of the parameter integral is not yet optimized for the purpose of renormalization. In particular, it shows a spurious singularity in the coefficient of the induced Maxwell term $\sim z^2$. This comes not unexpected as the cancellation of subdivergences implied by the Ward identity has, in a general gauge, no reason to be manifest at the parameter integral level.

We could improve on this either by switching to Landau gauge, or by performing a suitable partial integration on the integrand. The latter procedure is less systematic, but easy enough to implement for the simple case at hand: Inspection of the two versions which we have of this parameter integral, the original one eq.(9.48) and the partially integrated one eq.(9.51), shows that we can optimize the integrand by choosing a certain linear combination of both versions, namely

$$\mathcal{L}^{(2)}_{\rm scal}[B] = \frac{3}{4} \times \text{eq.(9.48)} + \frac{1}{4} \times \text{eq.(9.51)}. \tag{9.59}$$

(taking the photon insertion in Landau gauge would yield a similar simplification, though the resulting parameter integrals are not identical [32]). After integration over $\bar{T}$, this leads to another version of eq.(9.56),

$$\mathcal{L}^{(2)}_{\rm scal}[B] = -(4\pi)^{-4}\frac{e^2}{2}\int_0^\infty \frac{dT}{T^3} e^{-m^2T} \frac{z}{\sinh(z)} \int_0^1 du_a\, A'(z,u_a) \tag{9.60}$$

with a different integrand

$$\begin{aligned}
A' &= \Biggl\{ A'_0 \frac{\ln(G_{Bab}/G^z_{Bab})}{(G_{Bab} - G^z_{Bab})} + A'_1 \frac{\ln(G_{Bab}/G^z_{Bab})}{(G_{Bab} - G^z_{Bab})^2} \\
&\quad + \frac{A'_2}{(G^z_{Bab})(G_{Bab} - G^z_{Bab})} + \frac{A'_3}{(G_{Bab})(G_{Bab} - G^z_{Bab})} \Biggr\} \quad , \\
A'_0 &= 3\Bigl[2z^2 G^z_{Bab} - \frac{z}{\tanh(z)} - 1\Bigr] \\
A'_1 &= A_1 - \frac{3}{2}\Bigl[\dot{G}^2_{Bab} - \dot{G}^{z2}_{Bab}\Bigr] \\
A'_2 &= A_2 - \frac{3}{2}\Bigl[\dot{G}_{Bab}\dot{G}^z_{Bab} + \dot{G}^{z2}_{Bab}\Bigr] \\
A'_3 &= A_3 + \frac{3}{2}\Bigl[\dot{G}^2_{Bab} + \dot{G}_{Bab}\dot{G}^z_{Bab}\Bigr]
\end{aligned} \tag{9.61}$$

[32] We remark that they *would* be identical for the special case of a self-dual field.

We have not yet taken into account here the term involving δ_{ab}, stemming from $\ddot{\mathcal{G}}_{Bab}$, which was contained in the integrand of eq.(9.60). This term corresponds, in diagrammatic terms, to a tadpole insertion, and could therefore be safely deleted. However, it will be quite instructive to keep it and check explicitly that it is taken care of by the renormalization procedure. It leads to an integral $\int_0^\infty \frac{d\bar{T}}{\bar{T}^2}$ which we regulate by introducing an ultraviolet cutoff for the photon proper-time,

$$\int_{\bar{T}_0}^{\infty} \frac{d\bar{T}}{\bar{T}^2} = \frac{1}{\bar{T}_0} \tag{9.62}$$

It gives then a further contribution $E(\bar{T}_0)$ to $\mathcal{L}^{(2)}_{\rm scal}[B]$,

$$E(\bar{T}_0) = -3(4\pi)^{-4} e^2 \frac{1}{\bar{T}_0} \int_0^\infty \frac{dT}{T^2} e^{-m^2 T} \frac{z}{\sinh(z)} \tag{9.63}$$

Expanding the new integrand, $K'(z, u_a) \equiv \frac{z}{\sinh(z)} A'(z, u_a)$, in z, we find a much simpler result than before,

$$K'(z, u_a) = -6\frac{1}{G_{Bab}} + 3z^2 + O(z^4) \tag{9.64}$$

In particular, the absence of a subdivergence for the Maxwell term is now manifest.

We delete the irrelevant constant term, and add and subtract the Maxwell term. If we define

$$K_{02}(z, u_a) = -6\frac{1}{G_{Bab}} + 3z^2 \tag{9.65}$$

the Lagrangian becomes

$$\begin{aligned} \mathcal{L}^{(2)}_{\rm scal}[B] \quad = \quad & E(\bar{T}_0) - \frac{\alpha}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T}\, 3z^2 \\ & - \frac{\alpha}{2(4\pi)^3} \int_0^\infty \frac{dT}{T^3} e^{-m^2 T} \int_0^1 du_a \Big[K'(z, u_a) - K_{02}(z, u_a) \Big] \end{aligned} \tag{9.66}$$

The second term, which we denote by F, is divergent when integrated over the scalar proper-time T. We regulate it by introducing another proper-time cutoff T_0 for the scalar proper-time integral:

$$F(T_0) \quad := \quad -\frac{\alpha}{2(4\pi)^3} \int_{2T_0}^\infty \frac{dT}{T^3} e^{-m^2 T}\, 3z^2 \tag{9.67}$$

(compare [172]). The third term is convergent at $T = 0$, but still has a divergence at $u_a = u_b$, as it contains negative powers of G_{Bab}.

Expanding the integrand in a Laurent series in G_{Bab}, one finds

$$
\begin{aligned}
K'(z,u_a) - K_{02}(z,u_a) &= \frac{f(z)}{G_{Bab}} + O\Big(G^0_{Bab}\Big) \\
f(z) &= 3\Big[2 - \frac{z}{\sinh(z)} - \frac{z^2\cosh(z)}{\sinh(z)^2}\Big]
\end{aligned}
\tag{9.68}
$$

Again the singular part of this expansion is added and subtracted, yielding

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\rm scal}[B] &= E(\bar{T}_0) + F(T_0) - \frac{\alpha}{2(4\pi)^3}\int_{2T_0}^\infty \frac{dT}{T^3}e^{-m^2T}\int_{\frac{T_0}{T}}^{1-\frac{T_0}{T}} du_a\, \frac{f(z)}{G_{Bab}} \\
&\quad -\frac{\alpha}{2(4\pi)^3}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int_0^1 du_a\,\Big[K'(z,u_a) - K_{02}(z,u_a) - \frac{f(z)}{G_{Bab}}\Big]
\end{aligned}
\tag{9.69}
$$

The last integral is now completely finite. The third term, which we call $G(T_0)$, is finite at $T=0$, as $f(z) = O(z^4)$ by construction. Here we have introduced T_0 for the purpose of regulating the divergence at $u_a = u_b$.

Obviously this term cannot be made finite by photon wave function renormalization, so we must try to use the scalar mass renormalization for the purpose. This will be seen to work out in a quite nontrivial way. The u_a – integral for this term is readily computed and yields, in the limit $T_0 \to 0$, a contribution

$$
\int_{\frac{T_0}{T}}^{1-\frac{T_0}{T}} du_a \frac{1}{G_{Bab}} = -2\ln\Big(\frac{T_0}{T}\Big) = -2\ln(m^2T_0) + 2\ln(m^2T) \tag{9.70}
$$

We have rewritten this term for reasons which will become apparent in a moment. The upshot is that we can relate the function $f(z)$ to the scalar one-loop Euler-Heisenberg Lagrangian, eq.(5.23). If we write this Lagrangian for the pure magnetic field case, and subtract the two divergent terms lowest order in z, we obtain

$$
\bar{\mathcal{L}}^{(1)}_{\rm scal}[B] = \frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\mathrm{e}^{-m^2T}\Big[\frac{z}{\sinh(z)} + \frac{z^2}{6} - 1\Big] \tag{9.71}
$$

On the other hand, we can write

$$
\begin{aligned}
f(z) &= 3\Big[2 - \frac{z}{\sinh(z)} - \frac{z^2\cosh(z)}{\sinh(z)^2}\Big] \\
&= 3T^3\frac{d}{dT}\Big\{\frac{1}{T^2}\Big[\frac{z}{\sinh(z)} + \frac{z^2}{6} - 1\Big]\Big\}
\end{aligned}
\tag{9.72}
$$

By a partial integration over T, we can therefore re-express

$$
\begin{aligned}
\frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\mathrm{e}^{-m^2T}f(z) &= 3\frac{m^2}{(4\pi)^2}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr] \\
&= -3m^2\frac{\partial}{\partial m^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B]
\end{aligned}
\tag{9.73}
$$

(there are no boundary contributions since $f(z) = O(z^4)$). Next note that, at the two-loop level, the effect of mass renormalization consists in the following shift produced by the one-loop mass displacement δm^2,

$$
\delta\mathcal{L}^{(2)}_{\rm scal}[B] = \delta m^2\frac{\partial}{\partial m^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B]
\tag{9.74}
$$

To proceed, we need thus again the value of the one-loop mass displacement in scalar QED, but this time in proper-time regularization, and including its finite part. Again we give only the result,

$$
\delta m^2 = \frac{3\alpha}{4\pi}m^2\Bigl[-\ln\Bigl(m^2T_0\Bigr)-\gamma+c+\frac{1}{m^2\bar{T}_0}\Bigr]
\tag{9.75}
$$

where γ denotes the Euler-Mascheroni constant [33] , and c is a renormalization scheme dependent constant. Using this result, we may rewrite

$$
\begin{aligned}
G(T_0) &= \Bigl[\delta m^2 - 3c\frac{\alpha}{4\pi}m^2 - 3\frac{\alpha}{4\pi\bar{T}_0}\Bigr]\frac{\partial}{\partial m^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B] \\
&\quad -\frac{\alpha}{(4\pi)^3}\int_0^\infty\frac{dT}{T^3}\mathrm{e}^{-m^2T}\Bigl[\ln(m^2T)+\gamma\Bigr]f(z)
\end{aligned}
\tag{9.76}
$$

As expected the $\frac{1}{\bar{T}_0}$ – term introduced by the one-loop mass renormalization cancels the tadpole term $E(\bar{T}_0)$, up to its constant and Maxwell parts. Moreover, the complete divergence of $G(T_0)$ for $T_0 \to 0$ has been absorbed by δm^2. This is precisely the mechanism which had been found already in Ritus' analysis [247,244,246]. Putting all pieces together, we can write the complete two-loop approximation to the effective Lagrangian in the following way:

$$
\begin{aligned}
\mathcal{L}^{(\leq 2)}_{\rm scal}[B_0] &= -\frac{1}{2}B_0^2 - \frac{1}{(4\pi)^2}\int_{T_0}^\infty\frac{dT}{T^3}\mathrm{e}^{-m_0^2T}\frac{z^2}{6} + \bar{\mathcal{L}}^{(1)}_{\rm scal}[B_0] + \delta m_0^2\frac{\partial}{\partial m_0^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B_0] \\
&\quad -3c\frac{\alpha_0}{4\pi}m_0^2\frac{\partial}{\partial m_0^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B_0] - \frac{\alpha_0}{(4\pi)^3}\int_0^\infty\frac{dT}{T^3}\mathrm{e}^{-m_0^2T}\Bigl[\ln(m_0^2T)+\gamma\Bigr]f(z) \\
&\quad -\frac{\alpha_0}{2(4\pi)^3}\int_0^\infty\frac{dT}{T^3}e^{-m_0^2T}\int_0^1 du_a\,\Bigl[K'(z,u_a)-K_{02}(z,u_a)-\frac{f(z)}{G_{Bab}}\Bigr] \\
&\quad -\frac{\alpha_0}{2(4\pi)^3}\int_{2T_0}^\infty\frac{dT}{T^3}e^{-m_0^2T}\,z^2\Bigl(3-\frac{T}{\bar{T}_0}\Bigr)
\end{aligned}
\tag{9.77}
$$

[33] In comparing with [247,244,172,92] note that there this constant had been denoted by $\ln(\gamma)$.

We have rewritten this Lagrangian in bare quantities, since up to now we have been working in the bare regularized theory. Only mass and photon wave function renormalization are required to render this effective Lagrangian finite:

$$\begin{aligned} m^2 &= m_0^2 + \delta m_0^2 \\ e &= e_0 Z_3^{\frac{1}{2}} \\ B &= B_0 Z_3^{-\frac{1}{2}} \end{aligned} \tag{9.78}$$

(note that this leaves $z = e_0 B_0 T$ unaffected). Here δm_0^2 has already been introduced in eq.(9.75), while Z_3 is chosen so as to absorb the diverging one- and two-loop Maxwell terms in eq. (9.77). The final answer becomes

$$\begin{aligned} \mathcal{L}^{(\leq 2)}_{\text{scal}}[B] \;=\; & -\frac{1}{2}B^2 + \frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^3}\mathrm{e}^{-m^2T}\Big[\frac{z}{\sinh(z)} + \frac{z^2}{6} - 1\Big] \\ & +3c\frac{\alpha}{4\pi}m^2\frac{1}{(4\pi)^2}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\Big[\frac{z}{\sinh(z)} + \frac{z^2}{6} - 1\Big] \\ & -\frac{\alpha}{2(4\pi)^3}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int_0^1 du_a\,\Big[K'(z,u_a) - K_{02}(z,u_a) - \frac{f(z)}{G_{Bab}}\Big] \\ & -\frac{\alpha}{(4\pi)^3}\int_0^\infty \frac{dT}{T^3}\mathrm{e}^{-m^2T}\Big[\ln(m^2T) + \gamma\Big]f(z) \end{aligned} \tag{9.79}$$

e now denotes the physical charge. However, the result still contains the undetermined constant c, which appeared in the finite part of the one-loop scalar mass renormalization eq.(9.75). What remains to be done is to determine the value of c for which the renormalized mass m becomes the physical mass.

The worldline formalism applies, at least at the present stage of its development, only to the calculation of bare regularized amplitudes. As we have seen already in our β – function calculations, the renormalization of those amplitudes has to rely on auxiliary calculations in standard field theory. Usually worldline calculations are done in dimensional regularization, where there is no harm done in using different formalisms for the calculation of graphs and countergraphs, due to the universality of the regulator and the minimal subtraction prescription [34]. The same is not true for multiloop calculations using a proper-time cutoff, where one must make sure that the precise way of applying the cutoff is chosen consistently between graphs and countergraphs; otherwise one may have effectively performed unwanted finite renormalizations. This problem is well-known from multiloop calculations in scalar field theory performed with a naive momentum space cutoff (see [248] and refs. therein).

To ensure a correct identification of the physical scalar mass, we find it easiest to retrace the same calculation in dimensional regularization.

If one keeps the integrands in eqs. (9.48), (9.51) in D dimensions, instead of eq.(9.55) one obtains

[34] This may not hold in certain cases involving γ_5 or spacetime supersymmetry.

$$\begin{aligned}
\det{}^{-\frac{1}{2}}\Big[\frac{\sin(\mathcal{Z})}{\mathcal{Z}}(\bar{T}-\frac{1}{2}\mathcal{C}_{ab})\Big] &= \frac{z}{\sinh(z)}\gamma^{\frac{D}{2}-1}\gamma^{z}\\
\mathrm{tr}\Big[\ddot{\mathcal{G}}_{Bab}\Big] &= 2D\delta(\tau_a-\tau_b)-2(D-2)\frac{1}{T}-\frac{4}{T}\frac{z\cosh(z\dot{G}_{Bab})}{\sinh(z)}\\
\frac{1}{2}\mathrm{tr}\dot{\mathcal{G}}_{Bab}\mathrm{tr}\Big[\frac{\dot{\mathcal{G}}_{Bab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Big] &= \frac{1}{2}\Big[(D-2)\dot{G}_{Bab}+2\dot{G}^{z}_{Bab}\Big]\Big[(D-2)\dot{G}_{Bab}\gamma+2\dot{G}^{z}_{Bab}\gamma^{z}\Big]\\
\frac{1}{2}\mathrm{tr}\Big[\frac{(\dot{\mathcal{G}}_{aa}-\dot{\mathcal{G}}_{ab})(\dot{\mathcal{G}}_{ab}-\dot{\mathcal{G}}_{bb})}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Big] &= -\frac{1}{2}(D-2)\dot{G}^{2}_{Bab}\gamma-\Big[\dot{G}^{z2}_{Bab}+\frac{4}{T^2}z^2G^{z2}_{Bab}\Big]\gamma^{z} \qquad (9.80)
\end{aligned}$$

The term involving $\delta(\tau_a-\tau_b)$ can now be omitted, since in dimensional regularization it will not even contribute to the unrenormalized effective action (it leads to an integral $\int_0^\infty d\bar{T}\bar{T}^{-\frac{D}{2}}$ which vanishes according to the rules of dimensional regularization).

The linear combination eq.(9.59) generalizes to

$$\mathcal{L}^{(2)}_{\mathrm{scal}}[B]=\frac{D-1}{D}\times \mathrm{eq.}(9.48)+\frac{1}{D}\times\mathrm{eq.}(9.51) \qquad (9.81)$$

We rescale to the unit circle, $\tau_{a,b}=Tu_{a,b}$, set $\tau_b=0$ as usual, and also rescale $\bar{T}=T\hat{T}$. This yields an integral

$$\mathcal{L}^{(2)}_{\mathrm{scal}}[B]=-(4\pi)^{-D}\frac{e^2}{2}\int_0^\infty\frac{dT}{T}e^{-m^2T}T^{2-D}\int_0^\infty d\hat{T}\int_0^1du_a\,I(z,u_a,\hat{T},D) \qquad (9.82)$$

which is the $D-$dimensional version of eq.(9.60). Note that the rescaled integrand $I(z,u_a,\hat{T},D)$ depends on T only through z.

In contrast to the calculation in proper-time regularization, the $\hat{T}-$integration is nontrivial in dimensional regularization. It will therefore now be easier to extract all subdivergences *before* performing this integral. The analysis of the divergence structure shows that eqs.(9.64),(9.68) generalize to the dimensional case as follows,

$$K'(z,u_a,D)\equiv\int_0^\infty d\hat{T}\,I(z,u_a,\hat{T},D)=K_{02}(z,u_a,D)+f(z,D)G_{Bab}^{1-\frac{D}{2}}+O(z^4,G_{Bab}^{2-\frac{D}{2}}) \qquad (9.83)$$

with

$$\begin{aligned}
K_{02}(z,u_a,D) &= -4\frac{D-1}{D-2}G_{Bab}^{1-\frac{D}{2}}\\
&\quad+\frac{2}{3D(D-2)}\Big[(D-1)(D-4)G_{Bab}^{1-\frac{D}{2}}+(-2D^2+18D-4)G_{Bab}^{2-\frac{D}{2}}\Big]z^2\\
f(z,D) &= \frac{D-1}{D(D-2)}\Big[4D-\frac{2}{3}(D-4)z^2+(8-4D)\frac{z}{\sinh(z)}-8\frac{z^2\cosh(z)}{\sinh^2(z)}\Big]=O(z^4)
\end{aligned} \qquad (9.84)$$

After splitting off these two terms, the integral over the remainder is already finite, so that one can set $D=4$ in its computation. The $\hat{T}$ - integral then becomes elementary, and one is led back to eq.(9.61), since $K'(z,u_a,4)=K'(z,u_a)$.

Turning our attention to the second term on the right hand side of eq.(9.83), let us denote its contribution to the effective Lagrangian by $G_{\rm scal}(z,D)$,

$$G_{\rm scal}(z,D) = -(4\pi)^{-D}\frac{e^2}{2}\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{2-D}\int_0^1 du_a\, f(z,D)G_{Bab}^{1-\frac{D}{2}} \tag{9.85}$$

The equivalent of eq.(9.70) now reads

$$\int_0^1 du_a\, G_{Bab}^{1-\frac{D}{2}} = B\Bigl(2-\frac{D}{2},2-\frac{D}{2}\Bigr) = -\frac{4}{\epsilon}+0+O(\epsilon) \tag{9.86}$$

where B denotes the Euler Beta-function.

The identity eq.(9.72) generalizes to D dimensions as follows,

$$f(z,D) \quad = \quad 8\frac{D-1}{D(D-2)}T^{\frac{D}{2}+1}\frac{d}{dT}\Bigl\{T^{-\frac{D}{2}}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr]\Bigr\} \tag{9.87}$$

The partial integration over T now produces two terms,

$$\begin{aligned}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}T^{2-D}f(z,D) \quad &= \quad 8\frac{D-1}{D(D-2)}\Bigl\{m^2\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}T^{3-D}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr]\\ &+ \quad \frac{D-4}{2}\int_0^\infty \frac{dT}{T}\mathrm{e}^{-m^2T}T^{2-D}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr]\Bigr\}\end{aligned} \tag{9.88}$$

We now need the complete one-loop mass displacement calculated in dimensional regularization, which is [35]

$$\delta m^2 = m^2\frac{\alpha_0}{4\pi}\Bigl[-\frac{6}{\epsilon}+7-3[\gamma-\ln(4\pi)]-3\ln(m^2)\Bigr]+O(\epsilon) \tag{9.89}$$

Expanding eqs. (9.86), (9.88), and (9.74) in ϵ one finds that, up to terms of order $O(\epsilon)$,

$$\begin{aligned}G_{\rm scal}(z,D) \quad &= \quad \delta m^2\frac{\partial}{\partial m^2}\bar{\mathcal{L}}^{(1)}_{\rm scal}[B_0]+m^2\frac{\alpha_0}{(4\pi)^3}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr]\\ &\times\Bigl[-3\gamma-3\ln(m^2T)+\frac{3}{m^2T}+\frac{9}{2}\Bigr]\end{aligned} \tag{9.90}$$

Note that again the whole divergence of $G_{\rm scal}(z,D)$ for $D\to 4$ has been absorbed by δm^2.

Our final answer for the two-loop contribution to the finite renormalized scalar QED Euler-Heisenberg thus becomes

$$\begin{aligned}\mathcal{L}^{(2)}_{\rm scal}[B] \quad &= \quad -\frac{\alpha}{2(4\pi)^3}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int_0^1 du_a\,\Bigl[K'(z,u_a)-K_{02}(z,u_a)-\frac{f(z)}{G_{Bab}}\Bigr]\\ &+\frac{\alpha}{(4\pi)^3}m^2\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\Bigl[\frac{z}{\sinh(z)}+\frac{z^2}{6}-1\Bigr]\Bigl[-3\gamma-3\ln(m^2T)+\frac{3}{m^2T}+\frac{9}{2}\Bigr]\end{aligned} \tag{9.91}$$

[35] Note that this differs by a sign from δm^2 as used in eq.(9.30) - here δm^2 denotes the mass displacement itself, while there it denoted the corresponding counterterm.

This parameter integral representation is of a similar but simpler structure than the one obtained in [244,245], and we have not succeeded at a direct identification of both formulas. However, we have used MAPLE to expand both formulas in a Taylor expansion in B up to order $O(B^{20})$, and found exact agreement for the coefficients. Let us give the first few terms in this expansion,

$$\mathcal{L}^{(2)}_{\rm scal}[B] = \frac{\alpha m^4}{(4\pi)^3}\frac{1}{81}\biggl[\frac{275}{8}\Bigl(\frac{B}{B_{\rm cr}}\Bigr)^4 - \frac{5159}{200}\Bigl(\frac{B}{B_{\rm cr}}\Bigr)^6 + \frac{2255019}{39200}\Bigl(\frac{B}{B_{\rm cr}}\Bigr)^8 - \frac{931061}{3600}\Bigl(\frac{B}{B_{\rm cr}}\Bigr)^{10} + \dots\biggr] \tag{9.92}$$

The expansion parameter has been rewritten in terms of $B_{\rm cr} \equiv \frac{m^2}{e} \approx 4.4\cdot 10^{13} G$.

9.6.2. Spinor QED

The corresponding calculation for fermion QED is completely analogous, and we will present only the version in dimensional regularization.

In the superfield formalism, the formulas (9.48), (9.49) immediately generalize to the following integral representation for the two-loop effective action due to the spinor loop,

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm spin}[F] &= (-2)(4\pi)^{-D}\Bigl(-\frac{e^2}{2}\Bigr)\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar T\int_0^T d\tau_a d\tau_b \int d\theta_a d\theta_b\\
&\quad\times {\rm det}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\biggr]{\rm det}^{-\frac{1}{2}}\biggl[\bar T - \frac{1}{2}\hat{\mathcal{C}}_{ab}\biggr]\langle -D_a y_a\cdot D_b y_b\rangle\\
\langle -D_a y_a\cdot D_b y_b\rangle &= {\rm tr}\biggl[D_aD_b\hat{\mathcal{G}}_{ab} + \frac{1}{2}\frac{D_a(\hat{\mathcal{G}}_{aa}-\hat{\mathcal{G}}_{ab})D_b(\hat{\mathcal{G}}_{ab}-\hat{\mathcal{G}}_{bb})}{\bar T - \frac{1}{2}\hat{\mathcal{C}}_{ab}}\biggr]
\end{aligned} \tag{9.93}$$

Performing the Grassmann integrations, and removing $\ddot{\mathcal{G}}_{Bab}$ by partial integration over τ_a, we obtain the equivalent of eq. (9.51),

$$\begin{aligned}
\mathcal{L}^{(2)}_{\rm spin}[F] &= (4\pi)^{-D}e^2\int_0^\infty \frac{dT}{T}e^{-m^2T}T^{-\frac{D}{2}}\int_0^\infty d\bar T\int_0^T d\tau_a\int_0^T d\tau_b\\
&\quad\times {\rm det}^{-\frac{1}{2}}\biggl[\frac{\tan(\mathcal{Z})}{\mathcal{Z}}\Bigl(\bar T - \frac{1}{2}\mathcal{C}_{ab}\Bigr)\biggr]\frac{1}{2}\biggl\{{\rm tr}\dot{\mathcal{G}}_{Bab}{\rm tr}\biggl[\frac{\dot{\mathcal{G}}_{Bab}}{\bar T - \frac{1}{2}\mathcal{C}_{ab}}\biggr] - {\rm tr}\mathcal{G}_{Fab}{\rm tr}\biggl[\frac{\mathcal{G}_{Fab}}{\bar T - \frac{1}{2}\mathcal{C}_{ab}}\biggr]\\
&\quad+{\rm tr}\biggl[\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})(\dot{\mathcal{G}}_{Bab}-\dot{\mathcal{G}}_{Bbb}+2\mathcal{G}_{Faa}) + \mathcal{G}_{Fab}\mathcal{G}_{Fab} - \mathcal{G}_{Faa}\mathcal{G}_{Fbb}}{\bar T - \frac{1}{2}\mathcal{C}_{ab}}\biggr]\biggr\}
\end{aligned} \tag{9.94}$$

As we will discuss in more detail in the next section, in the spinor loop case this partially integrated integral is already a suitable starting point for renormalization.

Specializing to the magnetic field case, it is again easy to calculate the Lorentz determinants and traces. In addition to the bosonic ones calculated in eq.(9.80) we now need also

$$
\begin{aligned}
\mathrm{tr}\mathcal{G}_{Fab}\mathrm{tr}\Bigl[\frac{\mathcal{G}_{Fab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Bigr] &= G_{Fab}\Bigl[(D-2)+2\frac{\cosh(z\dot{G}_{Bab})}{\cosh(z)}\Bigr]G_{Fab}\Bigl[(D-2)\gamma+2\frac{\cosh(z\dot{G}_{Bab})}{\cosh(z)}\gamma^z\Bigr]\\
\mathrm{tr}\Bigl[\frac{(\dot{\mathcal{G}}_{Baa}-\dot{\mathcal{G}}_{Bab})\mathcal{G}_{Faa}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Bigr] &= 2\gamma^z\Bigl[1-\frac{\cosh(z\dot{G}_{Bab})}{\cosh(z)}\Bigr]\\
\mathrm{tr}\Bigl[\frac{\mathcal{G}_{Fab}\mathcal{G}_{Fab}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Bigr] &= (D-2)\gamma+2\gamma^z\frac{\cosh^2(z\dot{G}_{Bab})+\sinh^2(z\dot{G}_{Bab})}{\cosh^2(z)}\\
\mathrm{tr}\Bigl[\frac{\mathcal{G}_{Faa}\mathcal{G}_{Fbb}}{\bar{T}-\frac{1}{2}\mathcal{C}_{ab}}\Bigr] &= 2\tanh^2(z)\gamma^z
\end{aligned}
\tag{9.95}
$$

After rescaling to the unit circle, one obtains a parameter integral

$$
\mathcal{L}^{(2)}_{\mathrm{spin}}[B]=(4\pi)^{-D}e^2\int_0^\infty\frac{dT}{T}e^{-m^2T}T^{2-D}\int_0^\infty d\hat{T}\int_0^1du_a\,J(z,u_a,\hat{T},D) \tag{9.96}
$$

The extraction of the subdivergences yields

$$
L(z,u_a,D)\equiv\int_0^\infty d\hat{T}\,J(z,u_a,\hat{T},D)=L_{02}(z,u_a,D)+g(z,D)G_{Bab}^{1-\frac{D}{2}}+O(z^4,G_{Bab}^{2-\frac{D}{2}}) \tag{9.97}
$$

with

$$
L_{02}(z,u_a,D)=-4(D-1)G_{Bab}^{1-\frac{D}{2}}-\frac{4}{3D}\Bigl[(D-1)(D-4)G_{Bab}^{1-\frac{D}{2}}+(D-2)(D-7)G_{Bab}^{2-\frac{D}{2}}\Bigr]z^2 \tag{9.98}
$$

(compare eq.(9.25)), and

$$
g(z,D)=-\frac{4}{3}\frac{D-1}{D}\Bigl[6\frac{z^2}{\sinh^2(z)}+3(D-2)z\coth(z)-(D-4)z^2-3D\Bigr]=O(z^4) \tag{9.99}
$$

L_{02} is again removed by photon wave function renormalization. Denoting the contribution of the second term by $G_{\mathrm{spin}}(z,D)$, we note that the u_a - integral is the same as in the scalar QED case, eq. (9.86). Using the following identity analogous to eq. (9.72),

$$
g(z,D) = 8\frac{D-1}{D}T^{\frac{D}{2}+1}\frac{d}{dT}\Bigl\{T^{-\frac{D}{2}}\Bigl[\frac{z}{\tanh(z)}-\frac{z^2}{3}-1\Bigr]\Bigr\} \tag{9.100}
$$

we partially integrate the remaining integral over T. The $\frac{1}{\epsilon}$ - part of G_{spin} is then again found to be just right for absorbing the shift induced by the one-loop mass displacement,

$$
\delta m_0=m_0\frac{\alpha_0}{4\pi}\Bigl[-\frac{6}{\epsilon}+4-3[\gamma-\ln(4\pi)]-3\ln(m_0^2)\Bigr]+O(\epsilon) \tag{9.101}
$$

Up to terms of order ϵ one obtains

$$
\begin{aligned}
G_{\mathrm{spin}}(z,D) &= \delta m_0 \frac{\partial}{\partial m_0}\bar{\mathcal{L}}^{(1)}_{\mathrm{spin}}[B_0] + m_0^2 \frac{\alpha_0}{(4\pi)^3}\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m_0^2 T}\Big[\frac{z}{\tanh(z)} - \frac{z^2}{3} - 1\Big] \\
&\quad \times \Big[12\gamma + 12\ln(m_0^2 T) - \frac{12}{m_0^2 T} - 18\Big] \qquad (9.102)
\end{aligned}
$$

Our final result for the on-shell renormalized two-loop spinor QED Euler-Heisenberg Lagrangian is

$$
\begin{aligned}
\mathcal{L}^{(2)}_{\mathrm{spin}}[B] &= \frac{\alpha}{(4\pi)^3}\int_0^\infty \frac{dT}{T^3}e^{-m^2T}\int_0^1 du_a \Big[L(z,u_a,4) - L_{02}(z,u_a,4) - \frac{g(z,4)}{G_{Bab}}\Big] \\
&\quad - \frac{\alpha}{(4\pi)^3}m^2\int_0^\infty \frac{dT}{T^2}\mathrm{e}^{-m^2T}\Big[\frac{z}{\tanh(z)} - \frac{z^2}{3} - 1\Big]\Big[18 - 12\gamma - 12\ln(m^2T) + \frac{12}{m^2T}\Big] \\
&\qquad (9.103)
\end{aligned}
$$

with

$$
\begin{aligned}
L(z,u_a,4) &= \frac{z}{\tanh(z)}\Big\{B_1\frac{\ln(G_{Bab}/G^z_{Bab})}{(G_{Bab}-G^z_{Bab})^2} + \frac{B_2}{G^z_{Bab}(G_{Bab}-G^z_{Bab})} + \frac{B_3}{G_{Bab}(G_{Bab}-G^z_{Bab})}\Big\} \\
B_1 &= 4z\Big(\coth(z) - \tanh(z)\Big)G^z_{Bab} - 4G_{Bab} \\
B_2 &= 2\dot{G}_{Bab}\dot{G}^z_{Bab} + z(8\tanh(z) - 4\coth(z))G^z_{Bab} - 2 \\
B_3 &= 4G_{Bab} - 2\dot{G}_{Bab}\dot{G}^z_{Bab} - 4z\tanh(z)G^z_{Bab} + 2 \\
L_{02}(z,u_a,4) &= -\frac{12}{G_{Bab}} + 2z^2 \\
g(z,4) &= -6\Big[\frac{z^2}{\sinh(z)^2} + z\coth(z) - 2\Big] \qquad (9.104)
\end{aligned}
$$

Comparing with the previous results by Ritus [247,246] and Dittrich-Reuter [172], we have again not succeeded at a direct identification with the more complicated parameter integral given by Ritus. However, as in the scalar QED case we have verified agreement between both formulas up to the order of $O(B^{20})$ in the weak-field expansion in B. The first few coefficients are

$$
\mathcal{L}^{(2)}_{\mathrm{spin}}[B] = \frac{\alpha m^4}{(4\pi)^3}\frac{1}{81}\Big[64\Big(\frac{B}{B_{\mathrm{cr}}}\Big)^4 - \frac{1219}{25}\Big(\frac{B}{B_{\mathrm{cr}}}\Big)^6 + \frac{135308}{1225}\Big(\frac{B}{B_{\mathrm{cr}}}\Big)^8 - \frac{791384}{1575}\Big(\frac{B}{B_{\mathrm{cr}}}\Big)^{10} + \dots\Big] \quad (9.105)
$$

On the other hand, our formula *almost* allows for a term by term identification with the result of Dittrich-Reuter [172], as given in eqs. (7.21),(7.22),(7.37) there. This requires a rotation to Minkowskian proper-time, $T \to is$, a transformation of variables from u_a to $v := \dot{G}_{Bab}$, the use of trigonometric identities, and another partial integration over T for the second term in eq. (9.103). The only discrepancy arises in the constant 18, which reads 10 in the Dittrich-Reuter formula. One concludes that the results reached by Ritus and Dittrich-Reuter for the two-loop Euler-Heisenberg Lagrangian are incompatible, and differ precisely by a finite electron

mass renormalization [36]. Moreover, it is clearly Ritus' formula which correctly identifies the physical electron mass.

The corresponding result for the case of a pure electric field is obtained from this by the substitution $B^2 \rightarrow -E^2$. This sign chance makes an important difference, since it creates an imaginary part for the effective action, indicative of the possibility of pair creation in an electric field [169]. In [249] an approximation for this imaginary part was obtained from the weak field expansion for the magnetic case, using Borel summation and a dispersion relation.

The generalization of this calculation to the case of a general constant background field is straightforward [161].

Concerning the physical relevance of this type of calculation, let us mention the experiment PVLAS in preparation at Legnaro, Italy, which is an optical experiment designed to yield the first experimental measurement of the Euler-Heisenberg Lagrangian [250,251]. It is conceivable that the technology used there may even allow for the measurement of the two-loop correction in the near future [252,253].

9.7. Some More Remarks on the 2-Loop QED β – Functions

The calculation of the two-loop Euler-Heisenberg Lagrangian has to teach us also something about our previous β – function calculation. Of course, the β – function coefficients can be simply retrieved from the Euler-Heisenberg Lagrangians. Up to the contributions from mass renormalization, they can be read off from the expansions eq.(9.64), eq.(9.104)

$$
\begin{aligned}
K'(z, u_a, 4) &= -\frac{6}{G_{Bab}} + 3z^2 + O(z^4) \\
L(z, u_a, 4) &= -\frac{12}{G_{Bab}} + 2z^2 + O(z^4)
\end{aligned}
\tag{9.106}
$$

For example, the coefficient 2 appearing in the second line is nothing else but the -8 which we found in eq.(9.34) (up to the global factor of -2, and another factor of 2 which is due to the different choice of field strength tensors).

Comparing with that calculation we see that the use of the generalized Green's functions $\mathcal{G}_B, \mathcal{G}_F$ has saved us two integrations: The same formulas eq.(9.22) which there had been employed for executing the integrations over the points of interaction τ_1, τ_2 with the external field, have now entered already at the level of the construction of those Green's functions. Of course, for the β – function calculation all terms of order higher than $O(F^2)$ are irrelevant, so that one could then as well use the truncations of those Green's functions given in eq.(5.10). Moreover, the choice of an external field with the property $F^2 \sim \mathbb{1}$ is then more convenient than a magnetic field (this variant of the two-loop β – function calculation was presented in [93]).

More interestingly, we had noted before that, if the spinor QED β – function is calculated in a four-dimensional scheme, a subdivergence-free integrand is obtained proceeding directly from the partially integrated version eq.(9.94). We are now in a position to see that this fact is not accidental, but a consequence of renormalizability itself.

[36] The two formulas had been compared in [172] only in the strong field limit, which is not sensitive to this discrepancy.

Let us retrace our two-loop Euler-Heisenberg calculations, and analyze how the removal of all divergences worked for the Maxwell term. In the scalar QED case, there are three possible sources of quadratic divergences for the induced z^2 – term:

1. The contact term containing δ_{ab}.
2. The leading order term $\sim \frac{1}{G_{Bab}^2} z^2$ in the $\frac{1}{G_{Bab}}$ – expansion of the main term (see, e.g., eq.(9.58)).
3. The explicit $1/\bar{T}_0$ appearing in the one–loop mass displacement eq.(9.75).

The last one should cancel the other two in the renormalization procedure, if those are regulated by the same UV cutoff $\bar{T}_0$ for the photon proper-time, and this is indeed the case, as we verified in various versions of this calculation. In the spinor QED case the fermion propagator has no quadratic divergence (this is, of course, manifest in the standard first order formulation, while in the second order formulation discussed in section 4.11 there are several diagrams contributing to the electron self energy, and the absence of a quadratic divergence is due to a cancellation among them). The third term is thus missing, and the other two have to cancel among themselves. In particular, the completely partially integrated version of the integrand has no δ_{ab} – term any more, and consequently the second term must also be absent. But the structure of the integrals is such that, if one does this calculation in $D = 4$, the $\frac{1}{G_{Bab}}$ – expansion of the main contribution to the Maxwell term is always of the form shown in eq.(9.58),

$$\left[\frac{A}{G_{Bab}^2} + \frac{B}{G_{Bab}} + C\right] \mathrm{tr}(F^2) \tag{9.107}$$

with coefficients A, B, C. In the partially integrated version first the absence of a quadratic subdivergence allows one to conclude that $A = 0$, and then the absence of a logarithmic subdivergence that $B = 0$.

Note that this argument does not apply to the scalar QED case, nor does it to spinor QED in dimensional renormalization, due to the principal suppression of quadratic divergences by that scheme. In both cases one would have only one constraint equation for the two coefficients A and B appearing in the partially integrated integrand, and indeed they turn out to be nonzero in both cases. In the present formalism, the fermion QED two-loop β – function calculation thus becomes simpler when performed not in dimensional regularization, but in some four-dimensional scheme such as proper-time or Pauli-Villars regularization.

9.8. Beyond 2 Loops

This cancellation mechanism is interesting in view of some facts known about the three-loop fermion QED β – function [68,254,70]. Apart from the well-known cancellation of transcendental numbers occuring between diagrams in the calculation of the quenched (one fermion loop) contribution to this β – function [68,70], which takes place in any scheme and gauge, even more spectacular cancellations were found in [254] where this calculation was performed in four dimensions, Pauli-Villars regularization, and Feynman gauge. In that calculation all contributions from non-planar diagrams happened to cancel out exactly.

It appears that previously gauge invariance was considered as the only source of cancellations in this type of calculation. The cancellation mechanism which we exhibited in the previous

section is clearly of a different type, and specific to spinor QED. Whether this mechanism has a generalization to the three-loop level, or perhaps even relates to the cancellation found by Brandt, remains to be seen. In a preliminary study [255] we have computerized the generation of the partially integrated integrand for the three-loop spinor QED vacuum amplitude in a constant field, using MAPLE and M [256]. In the three-loop case the integrations over the proper-time parameters for the two inserted photons are still elementary. Therefore it turns out to be relatively easy to generate the unrenormalized 3-loop Euler-Heisenberg Lagrangians, which are now four-parameter integrals. However the analysis of the divergence structure along the above lines is a rather formidable problem, and so far no conclusive results have been reached. This is due not only to the large number of terms generated at the three-loop level, but also to the existence of several different subdivergences (in the notation of the example in section 8.5 those are at $a \sim b$, $c \sim d$, $(a,b) \sim (c,d)$, and $(a,b) \sim (d,c)$).

After discovering the rationality of the three - loop quenched QED β function many years ago, Rosner [68] conjectured that this β - function may perhaps provide a window to high orders in perturbation theory. Whether or not the worldline formalism will eventually allow one to go beyond the four orders presently accessible to other methods in this calculation, is impossible to say at present. Still we believe that this is a question very much worth pursuing, and that the answer will be a good indicator for the ultimate usefulness of this formalism.

10. Conclusions and Outlook

In this work we have reviewed the present status of the "string-inspired" technique, and its range of applications in perturbative quantum field theory. Although we have given a sketch of the original derivation of the Bern-Kosower rules from string theory, based on an analysis of the field theory limit of the string path integral, our overall emphasis has been on Strassler's more elementary "worldline" approach, using first-quantized particle path integral representations for one-loop effective actions. From our discussion of QCD amplitudes in chapter 4 it should have become clear that these two approaches complement each other nicely. The worldline approach provides a simple and efficient method for computing either the QCD effective action itself, or the one-particle irreducible N – gluon vertex function. The original string – based approach appears to be more powerful when it comes to the calculation of the N – gluon scattering amplitude. Here the worldline approach can still be used to correctly generate the input integrand for the Bern – Kosower rules, and also for a rederivation of the "loop replacement" part of those rules. However it presently still falls short of yielding a complete rederivation of the "pinch" part of the Bern – Kosower rules. On the other hand, the string – based approach is a priori restricted to the on-shell case, due to the requirement of conformal invariance. The off-shell continuation of string amplitudes in general leads to ambiguities, although at the one – loop level those seem now to be under control [72,73,75]. In particular, in [73] a method of continuation was given which in the field theory limit yields the gluon amplitudes in background field Feynman gauge. This line of work has recently led to the formulation of a general algorithm for computing the off-shell, one-loop multigluon amplitudes from the bosonic string [257].

Some complementarity holds also concerning the present range of applications of both methods. The string – based method has so far, apart from scalar amplitudes [72,76,77], essentially been applied only to one – loop gluon [26] and graviton [27,28,258] scattering amplitudes. The worldline path integral approach has provided a simple means to generate Bern – Kosower type master formulas also for field theories involving Yukawa [101,102] and axial [103,104,105,156] couplings. On the other hand, its application to gravitational backgrounds requires the mastery of some technical subtleties [60,61,62,144,225,226,227,228,229,230,231], as we discussed in section 7.2. In this context most authors have concentrated on the computation of anomalies [56,57,58,59,60,61,62] or of the effective action [224,226,230,231], although recently an interesting application was also given to the calculation of scattering amplitudes in eleven-dimensional supergravity [259].

Beyond the one – loop level, our presentation has been confined to the cases of scalar field theory and QED. As we already mentioned in the introduction, the construction of multiloop QCD amplitudes in the string – based formalism has turned out to be a formidable technical challenge [260]. Nevertheless, the recent progress in this line of work [72,73,74,75,76,77,78] seems to indicate that at least a computation of the two – loop QCD β – function may now be in reach. A parallel development using the worldline approach [89,95] has led to the derivation of a two – loop Euler-Heisenberg type action for pure Yang-Mills theory [261], albeit not yet in a form which would allow one to extract the two - loop Yang-Mills β – function.

As we have shown here, things are much easier in the abelian case. Here the worldline path integral approach provides an easy route to the construction of multiloop Bern-Kosower type formulas, based on the concept of multi-loop worldline Green's functions. Those Green's functions were explicitly constructed for Hamiltonian graphs, and carry the full information on

the internal propagators inserted into the Hamiltonian loop. Concerning the significance of this concept, from Roland and Sato's result eq.(8.32) one can see that this treatment of scalar propagator insertions is natural and "stringy". Whatever the ultimate form of the multiloop Bern-Kosower rules may turn out to be, it seems likely that those functions will figure in them prominently.

The same cannot be said for our treatment of photon insertions. Here it is rather clear that a truly string – based approach will incorporate internal photons in a more organic way. Nevertheless, our experience with the formalism at the two - and three - loop level clearly shows that it has many of the properties which one expects from a string - derived approach. It also seems to indicate that, at the multiloop level, quantum electrodynamics is a field theory which should be particularly suited to the application of string-derived techniques. This impression is based on the following properties:

- Particularly extensive cancellations are known to occur in multiloop QED calculations, suggesting that the standard field theory methods are far from optimized for this task.
- In contrast to scalar and non-abelian gauge theories, for QED amplitudes there exists a natural worldline parametrization exhausting the complete S-matrix.
- Sums of Feynman diagrams are always generated with the correct statistical weights.
- No colour factors exist which may prevent us from combining diagrams by "letting legs slide along lines".

In the presentation of this formalism given here we have tried to make maximal use of the worldline supersymmetry (which comes as a free gift from heaven in the worldline formalism). There are several aspects to this. The existence of this supersymmetry alone already leads to functional relationships between the parameter integrals for processes with the same external states, but different types of virtual particles involved. Moreover, the worldline superfield formalism allows one to treat scalar, spinor and gluon loops in a uniform manner, as well as to avoid the introduction of two different types of vertex operators.

The introduction of multiloop worldline Green's functions and of worldline superfields have a principle in common, which is that one should always try to absorb a maximum of information into the worldline propagators themselves. A third example of this principle was our introduction of worldline Green's functions incorporating constant external electromagnetic fields. As we have seen in section 9.5 those three instances of the principle can be freely combined with each other.

We have discussed a large number of applications, often in considerable detail. Some of those calculations have been of an illustrative nature, or mere consistency checks. Others were state-of-the-art calculations, such as our recalculations of the QED photon splitting amplitudes and of the two-loop Euler-Heisenberg Lagrangians. Those examples clearly display the technical advantages over standard field theory techniques which one can hope to achieve in this formalism, at least for certain types of calculations. However, they also show that the string – inspired technique is presently still a rather specialized tool. All of our applications have been to processes involving a loop *with no change of the identity of the particle inside the loop.* This class of amplitudes seems to be the most natural one to consider in this context, although it is by no means the only one to which string – inspired techniques can be applied. Notably

a generalization of the Bern-Kosower formalism to amplitudes involving external quarks was constructed by L. Dixon [262], but in preliminary studies has not led to as significant an improvement over field theory methods as was found in the calculation of the four and five gluon amplitudes [263] [37]. Similarly also the worldline formalism can be easily extended to the computation of fermion self energies [219,266,267]. However the resulting formalism seems somewhat less elegant than in the photon self energy case, and has been too little explored yet to be presented here.

Another omission in the present review is the extension to the finite temperature case, which has been considered by various authors [268,269,270,271,184,272]. The most general result which has been reached in this line of work is a generalization of the QED Bern – Kosower master formula to the N – photon amplitude at finite temperature and chemical potential [272]. However, this result holds for the Euclidean amplitude, and for most physical applications would still have to be analytically continued. Here one encounters the same type of ambiguities as in Feynman parameter calculations at finite temperature [273,274], which makes it presently difficult to judge the practical usefulness of this generalization.

Finally, it is quite possible that the use made of worldline path integrals in the present work may appear overly modest to the enterprising reader. Our whole aim here was to reproduce the S-matrices of known renormalizable field theories in a way which avoids some of the shortcomings of conventional Feynman diagram calculations. Clearly it is always possible to rewrite a given field theory amplitude in terms of worldline path integrals, although not in all cases this will lead to calculational improvements. Much less obvious is the converse question, which is whether a "sensible" worldline Lagrangian must always be induced by a spacetime Lagrangian, or whether worldline path integrals can perhaps be used to define physically relevant S-matrices which do not correspond to Lagrangian field theories. The use of an additional worldline curvature term [275,276,277] could be seen as a step in this direction.

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[37] This evaluation could possibly change due to recent progress [264,265] with regard to the computation of heterotic string amplitudes involving external fermions.

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Appendix A. Summary of Conventions

At the path integral level, we work in the Euclidean throughout with a positive definite metric $(g_{\mu\nu}) = \mathrm{diag}(++\ldots+)$. Our Euclidean Dirac matrix conventions are

$$\{\gamma_\mu, \gamma_\nu\} \;=\; 2g_{\mu\nu}\mathbb{1}, \quad \gamma_\mu^\dagger = \gamma_\mu, \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \quad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] \tag{A.1}$$

The Euclidean field strength tensor is defined by $F^{ij} = \varepsilon_{ijk}B_k, i,j = 1,2,3,\ F^{4i} = -iE_i$, its dual by $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F^{\alpha\beta}$ with $\varepsilon^{1234} = 1$.

The corresponding Minkowski space amplitudes are obtained by replacing

$$\begin{aligned} g_{\mu\nu} &\rightarrow \eta_{\mu\nu} \\ k^4 &\rightarrow -ik^0 \\ T &\rightarrow is \\ \varepsilon^{1234} &\rightarrow i\varepsilon^{1230} \\ F^{4i} &\rightarrow F^{0i} = E_i \\ \tilde{F}^{\mu\nu} &\rightarrow -i\tilde{F}^{\mu\nu} \end{aligned} \tag{A.2}$$

where $(\eta_{\mu\nu}) = \mathrm{diag}(-+++), \quad \varepsilon^{0123} = 1$.

The non-abelian covariant derivative is $D_\mu \equiv \partial_\mu + igA_\mu^a T^a$, with $[T^a, T^b] = if^{abc}T^c$. The adjoint representation is given by $(T^a)^{bc} = -if^{abc}$, and the generators in the fundamental representation of $SU(N_c)$ are normalized as $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

Momenta appearing in vertex operators are *ingoing*.

Appendix B. Worldline Green's Functions

In this appendix we derive the generalized worldline Green's functions $G^C_{P,A}$ needed for the evaluation of the gluon – path integral (section 4.6) and $\mathcal{G}_{B,F}$ for the scalar/spinor path integral in a constant external field (chapter 5).

All those Green's functions are kernels of certain integral operators, acting in the real Hilbert space of periodic or antiperiodic functions defined on an interval of length T. We denote by $\bar{H}_P$ the full space of periodic functions, by H_P the same space with the constant mode exempted, and by H_A the space of antiperiodic functions. The ordinary derivative acting on those functions is correpondingly denoted by ∂_P, $\bar{\partial}_P$ or ∂_A. With those definitions, we can write our Green's functions as

$$
\begin{aligned}
\mathcal{G}_B(\tau_1,\tau_2) &= 2\langle\tau_1 \mid \Bigl(\partial_P{}^2 - 2iF\partial_P\Bigr)^{-1} \mid \tau_2\rangle \\
\mathcal{G}_F(\tau_1,\tau_2) &= 2\langle\tau_1 \mid \Bigl(\partial_A - 2iF\Bigr)^{-1} \mid \tau_2\rangle \\
G^C_P(\tau_1,\tau_2) &= \langle\tau_1 \mid \Bigl(\bar{\partial}_P - C\Bigr)^{-1} \mid \tau_2\rangle \\
G^C_A(\tau_1,\tau_2) &= \langle\tau_1 \mid \Bigl(\partial_A - C\Bigr)^{-1} \mid \tau_2\rangle
\end{aligned}
\tag{B.1}
$$

(in this appendix we absorb the coupling constant e into the external field F). Note that G^C_A is, up to a conventional factor of 2, formally identical with $\mathcal{G}_F$ under the replacement $C \to 2iF$.

$\mathcal{G}_B$ and $\mathcal{G}_F$ are easy to construct using the following representation of the integral kernels for inverse derivatives [85]

$$\langle u \mid \partial_P{}^{-n} \mid u'\rangle = -\frac{1}{n!}B_n(\mid u-u' \mid)\mathrm{sign}^n(u-u') \tag{B.2}$$

$$\langle u \mid \partial_A{}^{-n} \mid u'\rangle = \frac{1}{2(n-1)!}E_{n-1}(\mid u-u' \mid)\mathrm{sign}^n(u-u') \tag{B.3}$$

Here $B_n(E_n)$ denotes the n^{th} Bernoulli (Euler) polynomial, and we have set $T=1$. Those formulas are valid for $\mid u-u' \mid \leq 1$. Let us shortly prove the first identity; the proof of the second one is completely analogous.

First observe that, by construction, $\frac{1}{2}\dot{G}_B$ is the integral kernel inverting the first derivative ∂_P acting on periodic functions. We may therefore write

$$
\begin{aligned}
K_n(u_1-u_{n+1}) &:= \int_0^1 du_2\ldots du_n \dot{G}_{B12}\dot{G}_{B23}\ldots\dot{G}_{Bn(n+1)} \\
&= 2^n < u_1 \mid \partial_P{}^{-n} \mid u_{n+1} >
\end{aligned}
\tag{B.4}
$$

This leads to the recursion relation

$$\frac{\partial}{\partial u} K_n(u-u') = 2^n < u \mid \partial_P{}^{-(n-1)} \mid u' >= 2K_{n-1}(u-u') \tag{B.5}$$

We want to show that the same recursion relation is fulfilled by the polynomial $\tilde{K}_n$,

$$\tilde{K}_n(u-u') := -\frac{2^n}{n!} B_n(|u-u'|)\mathrm{sign}^n(u-u') \tag{B.6}$$

Explicit differentiation yields

$$\begin{aligned} \frac{\partial}{\partial u}\tilde{K}_n(u-u') &= -\frac{2^n}{n!} B_n'(|u-u'|)\mathrm{sign}(u-u')\mathrm{sign}^n(u-u') \\ &= -\frac{2^n}{(n-1)!} B_{n-1}(|u-u'|)\mathrm{sign}^{n+1}(u-u') \\ &= 2\tilde{K}_{n-1}(u-u') \end{aligned} \tag{B.7}$$

Here the recursion relation for the Bernoulli polynomials was used, $\frac{d}{dx}B_n(x) = nB_{n-1}(x)$. An additional term arising by differentiation of the signum function for n odd can be deleted due to the fact that

$$\delta(x)B_n(\mid x \mid) = \delta(x)B_n(0) = 0 \tag{B.8}$$

for n odd, $n > 1$. The proof is completed by checking that the master identity works for $n = 1$ ($B_1(x) = x - \frac{1}{2}$), and on the diagonal $u_1 = u_{n+1}$ for any n. The second statement is trivial for odd n, since here both sides vanish by antisymmetry. For even n it becomes

$$Tr(\partial_P^{-n}) \quad = \quad -\frac{B_n}{n!} \tag{B.9}$$

with $B_n = B_n(0)$ the n^{th} Bernoulli number. This identity is easily shown by writing the trace in the eigenbasis $\{\mathrm{e}^{2\pi iku}, k \in \mathbf{Z}\backslash\{0\}\}$, and using the well-known relation between the Bernoulli numbers and the values of the Riemann ζ - function at positive even numbers, $\zeta(2n) = \frac{(2\pi)^{2n}}{2(2n)!}|B_{2n}|$.

Using (B.2), we can compute $\mathcal{G}_B$ for the unit circle as follows:

$$\begin{aligned} \mathcal{G}_B(u_1,u_2) &= 2\langle u_1 \mid \Bigl(\partial_P{}^2 - 2iF\partial_P\Bigr)^{-1} \mid u_2\rangle \\ &= 2\sum_{n=0}^{\infty}(2iF)^n\langle u_1|\partial_P{}^{-(n+2)}|u_2\rangle \\ &= -2\sum_{n=2}^{\infty}\frac{(2iF)^{n-2}\mathrm{sign}^n(u_1-u_2)}{n!}B_n(\mid u_1-u_2\mid) \\ &= -\frac{1}{iF}\frac{\mathrm{sign}(u_1-u_2)\mathrm{e}^{2iF(u_1-u_2)}}{\mathrm{e}^{2iF\mathrm{sign}(u_1-u_2)}-1} + \frac{\mathrm{sign}(u_1-u_2)}{iF}B_1(\mid u_1-u_2\mid) - \frac{1}{2F^2} \\ &= \frac{1}{2F^2}\biggl(\frac{F}{\mathrm{sin}F}\mathrm{e}^{-iF\dot{G}_{B12}} + iF\dot{G}_{B12} - 1\biggr) \end{aligned} \tag{B.10}$$

In the next-to-last step we used the generating identity for the Bernoulli polynomials,

$$\frac{t\mathrm{e}^{xt}}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} \tag{B.11}$$

This is $\mathcal{G}_B$ as given in eq.(5.6) up to a simple rescaling. The computation of $\mathcal{G}_F$ proceeds in a completely analogous way.

This method does not work for the determination of G_P^C, since negative powers of $\bar{\partial}_P$ are not even well-defined in the presence of the zero mode. In the following, we will calculate $G_{P,A}^C$ in a different way, which corresponds to the usual construction of the Feynman propagator in field theory. In order to determine $G_A^C(\tau)$, say, we employ the following set of basis functions over the circle with circumference T:

$$f_n(\tau) = T^{-1/2}\exp\Big[i\frac{2\pi}{T}(n+\frac{1}{2})\tau\Big], \quad n \in Z \tag{B.12}$$

They satisfy

$$\begin{aligned} &\int_0^T d\tau f_n^*(\tau)f_m(\tau) = \delta_{nm} \\ &\sum_{n=-\infty}^{\infty} f_n(\tau_2)f_n^*(\tau_1) = \sum_{m=-\infty}^{\infty}\delta(\tau_2-\tau_1-mT) \end{aligned} \tag{B.13}$$

and $f_n(\tau+T) = -f_n(\tau)$. In this basis, the Green's function (4.49) becomes

$$G_A^C(\tau_1,\tau_2) = G_A^C(\tau_1-\tau_2) = \frac{1}{T}\sum_{n=-\infty}^{\infty}\frac{\exp\Big[i\frac{2\pi}{T}(n+\frac{1}{2})(\tau_1-\tau_2)\Big]}{i(2\pi/T)(n+\frac{1}{2})-C} \tag{B.14}$$

By introducing an auxiliary integration in the form ($\tau \equiv \tau_1 - \tau_2$)

$$G_A^C(\tau) = \int_{-\infty}^{\infty} d\omega\,\frac{1}{T}\sum_{n=-\infty}^{\infty}\delta\Big(\omega - \frac{2\pi}{T}(n+\frac{1}{2})\Big)\frac{e^{i\omega\tau}}{i\omega - C} \tag{B.15}$$

and using Poisson's resummation formula, the Green's function assumes the suggestive form [119]

$$G_A^C(\tau) = \sum_{n=-\infty}^{\infty}(-1)^n G_\infty^C(\tau+nT) \tag{B.16}$$

with

$$G_\infty^C(\tau) \equiv \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{i\omega\tau}}{i\omega - C} \tag{B.17}$$

We verify that

$$\Big(\frac{d}{d\tau} - C\Big)G_\infty^C(\tau) = \delta(\tau) \tag{B.18}$$

$$\Big(\frac{d}{d\tau} - C\Big)G_A^C(\tau) = \sum_{n=-\infty}^{\infty}(-1)^n\delta(\tau+nT) \tag{B.19}$$

which shows that G_∞^C is a Green's function on the infinitely extended real line, while G_A^C is defined on the circle. The integral (B.17) yields for $C > 0$

$$G_\infty^C(\tau) = -\theta(-\tau)e^{C\tau} \tag{B.20}$$

Hence, from (B.16)

$$G_A^C(\tau) = -e^{C\tau} \sum_{n=-\infty}^{\infty} (-1)^n \theta(-\tau - nT) e^{nCT} \tag{B.21}$$

For $\tau \in (0,T)$ only the terms $n = -\infty, ..., -1$ contribute to the sum in (B.21), while for $\tau \in (-T, 0)$ a nonzero contribution is obtained for $n = -\infty, ..., 0$. Summing up the geometric series in either case and combining the results we obtain the expression given in eq. (4.49). It is valid for $-T < \tau < +T$. Using a basis of periodic functions the same arguments lead to G_P^C as stated in (4.48). Note that in the limit of a large period T

$$\lim_{T\to\infty} G_{A,P}^C(\tau) = G_\infty^C(\tau) \tag{B.22}$$

as it should be. For $C \to 0$, both G_∞^C and G_A^C have a well-defined limit:

$$\begin{aligned} G_\infty^0(\tau) &= -\theta(-\tau) \\ G_A^0(\tau) &= \frac{1}{2} \text{ sign } (\tau) \end{aligned} \tag{B.23}$$

The periodic Green's function G_P^C blows up in this limit because $\bar{\partial}_P^{-1}$ does not exist in presence of the constant mode. It is important to keep in mind that G_P^C is defined in such a way that it includes the zero mode of $\frac{d}{d\tau}$.

In the perturbative evaluation of the spin-1 path integral one has to deal with traces over chains of propagators of the form

$$\sigma_{A,P}^n(C) \equiv \text{Tr}_{A,P}\Big[(\frac{d}{d\tau} - C)^{-n}\Big] \tag{B.24}$$

Because

$$\sigma_{A,P}^n(C) = \frac{1}{(n-1)!}\left(\frac{d}{dC}\right)^{n-1} \sigma_{A,P}^1(C) \tag{B.25}$$

it is sufficient to know $\sigma_{A,P}^1(C)$. The subtle point which we would like to mention here is that strictly speaking the sum defining σ_A^1, say,

$$\sigma_A^1(C) = \sum_{n=-\infty}^{\infty} \frac{1}{i(2\pi/T)(n+\frac{1}{2}) - C} \tag{B.26}$$

does not converge as it stands, and is meaningless without a prescription of how to regularize it. The usual strategy is to combine terms for positive and negative values of n, and to replace (B.26) by the convergent series

$$\begin{aligned} \sigma_A^1(C) &= -2C \sum_{n=0}^{\infty} \left[\left(\frac{2\pi}{T}\right)^2 (n+\frac{1}{2})^2 + C^2\right]^{-1} \\ &= -\frac{T}{2} \tanh\left(\frac{CT}{2}\right) \end{aligned} \tag{B.27}$$

It is important to realize that this definition implies a well-defined prescription for the treatment of the θ functions in $G_{A,P}^C$ at $\tau = 0$. In fact,

$$\sigma_A^1(C) = \int_0^T d\tau \; G_A^C(\tau - \tau) = T \; G_A^C(0) \tag{B.28}$$

and by combining eqs. (B.27) and (B.28) we deduce that we must set

$$\lim_{\tau \searrow 0} \theta(\tau) = \lim_{\tau \searrow 0} \theta(-\tau) = \frac{1}{2} \tag{B.29}$$

With (B.27) we obtain

$$\sigma_A^n(C) = -\frac{1}{(n-1)!} \left(\frac{T}{2}\right)^n \left(\frac{d}{dx}\right)^{n-1} \tanh(x)\Big|_{x=CT/2} \tag{B.30}$$

The analogous relation in the periodic case is

$$\sigma_P^n(C) = -\frac{1}{(n-1)!} \left(\frac{T}{2}\right)^n \left(\frac{d}{dx}\right)^{n-1} \coth(x)\Big|_{x=CT/2} \tag{B.31}$$

if the zero mode of $\frac{d}{d\tau}$ is included in the trace (B.24), and

$$\sigma'^n_P(C) = -\frac{1}{(n-1)!} \left(\frac{T}{2}\right)^n \left(\frac{d}{dx}\right)^{n-1} \{\coth(x) - x^{-1}\}\Big|_{x=CT/2} \tag{B.32}$$

if the zero mode is omitted. For C sufficiently small one finds the power series expansions

$$\begin{aligned}
\sigma_A^n(C) &= -\frac{1}{(n-1)!} \sum_{k=n/2}^{\infty} \frac{(2^{2k}-1)B_{2k}}{2k(2k-n)!} T^{2k} C^{2k-n} \\
\sigma'^n_P(C) &= -\frac{1}{(n-1)!} \sum_{k=n/2}^{\infty} \frac{B_{2k}}{2k(2k-n)!} T^{2k} C^{2k-n}
\end{aligned} \tag{B.33}$$

σ_A^n and σ'^n_P have well defined limits for $C \to 0$:

$$\begin{aligned}
\sigma_A^n(0) &= -\frac{(2^n-1)B_n}{n!} T^n = \frac{1}{2}\frac{E_{n-1}(0)}{(n-1)!} T^n \quad (n \text{ even}) \\
\sigma'^n_P(0) &= -\frac{B_n}{n!} T^n \quad (n \text{ even})
\end{aligned} \tag{B.34}$$

(those limits vanish for n odd). This brings us, of course, back to eqs.(B.2), (B.3).

Appendix C. Symmetric Partial Integration

In this appendix we explain a partial integration algorithm which allows one to remove all $\ddot{G}_{Bij}$'s contained in the original numerator polynomial P_N (4.68) of the N - photon amplitude, and which preserves the full permutation symmetry in the N photons.

Such an "impartial" partial integration algorithm can be defined in the following way:

1. In every step, partially integrate away *all* $\ddot{G}_{Bij}$'s appearing in the term under inspection *simultaneously*. This is possible since different $\ddot{G}_{Bij}$'s do not share variables to being with, and this property is preserved by all partial integrations. New $\ddot{G}_{Bij}$'s may be created.
2. In the first step, for every $\ddot{G}_{Bij}$ partially integrate both over τ_i and τ_j, and take the mean of the results.
3. At every following step, any $\ddot{G}_{Bij}$ appearing must have been created in the previous step. Therefore either both i and j were used in the previous step, or just one of them. If both, the rule is to again use both variables in the actual step for partial integration, and take the mean of the results. If only one of them was used in the previous step, then the other one should be used in the actual step.

For example, the term $\ddot{G}_{B12}\ddot{G}_{B34}$ appearing in P_4 in the first step transforms as follows,

$$\begin{aligned}\ddot{G}_{B12}\ddot{G}_{B34} \quad \rightarrow \quad & \frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\Big\{\Big[\dot{G}_{B1i}k_1\cdot k_i - \dot{G}_{B2i}k_2\cdot k_i\Big]\Big[\dot{G}_{B3j}k_3\cdot k_j - \dot{G}_{B4j}k_4\cdot k_j\Big] \\ & -\ddot{G}_{B13}k_1\cdot k_3 + \ddot{G}_{B14}k_1\cdot k_4 + \ddot{G}_{B23}k_2\cdot k_3 - \ddot{G}_{B24}k_2\cdot k_4\Big\}\end{aligned} \tag{C.1}$$

The terms in the second line have to be further processed. Considering just the first one of them, since both variables appearing in $\ddot{G}_{B13}$ were active in the first step, both must also be used in the second one. This yields

$$\begin{aligned}-\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\ddot{G}_{B13} \quad \rightarrow \quad & \frac{1}{8}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}\Big[\dot{G}_{B1i}k_1\cdot k_i - \dot{G}_{B3i}k_3\cdot k_i\Big] \\ & +\frac{1}{8}\dot{G}_{B13}\Big[\ddot{G}_{B12}\dot{G}_{B34} - \dot{G}_{B12}\ddot{G}_{B34}\Big]\end{aligned} \tag{C.2}$$

Considering again the first term in the second line, only τ_1 was active in the previous step. Therefore only τ_2 must be used now, and the third step is the final one,

$$\frac{1}{8}\dot{G}_{B13}\ddot{G}_{B12}\dot{G}_{B34} \rightarrow \frac{1}{8}\dot{G}_{B13}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B2i}k_2\cdot k_i \tag{C.3}$$

This prescription treats all variables on the same footing, and therefore must lead to a permutation symmetric result. The nontrivial fact is that the process terminates after a finite

number of steps, and does not become cyclic (as would be the case if, for example, one would *always* treat the indices in a $\ddot{G}_{Bij}$ symmetrically). This is not difficult to derive from the fact that, for any term in P_N, the indices appearing in the $\ddot{G}_{Bij}$'s and the first indices of the $\dot{G}_{Bij}$'s are associated to the polarization vectors, and thus must all take different values.

This algorithm transforms P_4 into

$$
\begin{aligned}
Q_4 \;=\;& \dot{G}_{B1i}\varepsilon_1\cdot k_i\dot{G}_{B2j}\varepsilon_2\cdot k_j\dot{G}_{B3k}\varepsilon_3\cdot k_k\dot{G}_{B4l}\varepsilon_4\cdot k_l\\
&+\Biggl\{\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\Biggl\{\dot{G}_{B3i}\varepsilon_3\cdot k_i\dot{G}_{B4j}\varepsilon_4\cdot k_j\Bigl[\dot{G}_{B1k}k_1\cdot k_k-\dot{G}_{B2k}k_2\cdot k_k\Bigr]\\
&\quad+\Bigl[\dot{G}_{B3i}\varepsilon_3\cdot k_i(\dot{G}_{B41}\varepsilon_4\cdot k_1-\dot{G}_{B42}\varepsilon_4\cdot k_2)\dot{G}_{B4k}k_4\cdot k_k+(3\leftrightarrow 4)\Bigr]\\
&\quad+\Bigl[(\dot{G}_{B31}\varepsilon_3\cdot k_1-\dot{G}_{B32}\varepsilon_3\cdot k_2)\dot{G}_{B43}\varepsilon_4\cdot k_3\dot{G}_{B4k}k_4\cdot k_k+(3\leftrightarrow 4)\Bigr]\Biggr\}+5\text{ permutations}\Biggr\}\\
&+\Biggl\{\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Biggl\{\Bigl[\dot{G}_{B1i}k_1\cdot k_i-\dot{G}_{B2i}k_2\cdot k_i\Bigr]\Bigl[\dot{G}_{B3j}k_3\cdot k_j-\dot{G}_{B4j}k_4\cdot k_j\Bigr]\\
&\quad+\frac{1}{2}\Bigl[\dot{G}_{B13}k_1\cdot k_3-\dot{G}_{B23}k_2\cdot k_3-\dot{G}_{B14}k_1\cdot k_4+\dot{G}_{B24}k_2\cdot k_4\Bigr]\\
&\quad\times\Bigl[\dot{G}_{B1i}k_1\cdot k_i+\dot{G}_{B2i}k_2\cdot k_i-\dot{G}_{B3i}k_3\cdot k_i-\dot{G}_{B4i}k_4\cdot k_i\Bigr]\Biggr\}+2\text{ perm.}\Biggr\}
\end{aligned}
\tag{C.4}
$$

This expression can be rewritten more compactly as follows,

$$
Q_4=q_4^4+q_4^3+q_4^2-q_4^{22}
\tag{C.5}
$$

where

$$
\begin{aligned}
q_4^4 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B34}\dot{G}_{B41}Z_4(1234)+2\text{ permutations}\\
q_4^3 &= \dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\dot{G}_{B4i}\varepsilon_4\cdot k_i+3\text{ perm.}\\
q_4^2 &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\Bigl\{\dot{G}_{B3i}\varepsilon_3\cdot k_i\dot{G}_{B4j}\varepsilon_4\cdot k_j+\frac{1}{2}\dot{G}_{B34}\varepsilon_3\cdot\varepsilon_4\Bigl[\dot{G}_{B3i}k_3\cdot k_i-\dot{G}_{B4i}k_4\cdot k_i\Bigr]\Bigr\}\\
&\quad+5\text{ perm.}\\
q_4^{22} &= \dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B34}\dot{G}_{B43}Z_2(34)+2\text{ perm.}
\end{aligned}
\tag{C.6}
$$

and the “Lorentz cycles” Z_n,

$$
\begin{aligned}
Z_2(ij) &\equiv \varepsilon_i\cdot k_j\varepsilon_j\cdot k_i-\varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\\
Z_n(i_1i_2\dots i_n) &\equiv \operatorname{tr}\prod_{j=1}^{n}\Bigl[k_{i_j}\otimes\varepsilon_{i_j}-\varepsilon_{i_j}\otimes k_{i_j}\Bigr]\quad(n\geq 3)
\end{aligned}
$$

have already been introduced in (4.70). Note that the product of two-cycles q_4^{22} appears with a minus sign in eq.(C.5). The reason is that we corrected for an over-counting here; q_4^{22} is also contained twice in q_4^2, and separating it out from there will change the “-” to a “+”.

Before proceeding to higher point amplitudes, it will be prudent to further condense the notation. We thus abbreviate

$$
\begin{aligned}
\dot{G}_{ij} &\equiv \dot{G}_{Bij}\varepsilon_i \cdot k_j \\
\underline{\dot{G}}_{ij} &\equiv \dot{G}_{Bij}\varepsilon_i \cdot \varepsilon_j \\
\dot{\not{G}}_{ij} &\equiv \dot{G}_{Bij} k_i \cdot k_j \\
\dot{G}(i_1 i_2 \dots i_n) &\equiv \dot{G}_{Bi_1 i_2}\dot{G}_{Bi_2 i_3}\cdots \dot{G}_{Bi_n i_1} Z_n(i_1 i_2 \dots i_n)
\end{aligned}
\tag{C.7}
$$

As was already mentioned in the main text, it is known from previous work [20,21,150] that a closed "τ – cycle" $\dot{G}_{Bi_1 i_2}\dot{G}_{Bi_2 i_3}\cdots \dot{G}_{Bi_n i_1}$ after the partial integration will always appear multiplied by a factor of $Z_n(i_1 i_2 \dots i_n)$. This motivates the last one of the abbreviations above, and also explains why the formulation of the "cycle substitution" part of the Bern-Kosower rules did not require the specification of a particular partial integration algorithm.

A given term in Q_N thus will be a product of "bi-cycles" $\dot{G}(\cdot)$, multiplied by a remainder. Following [150] we call this remainder "tail", or "m - tail", where m denotes the number of indices not appearing in any of the cycles. For example, q_4^2 is the product of a 2 – bi-cycle and a 2 - tail. Only the tails depend on the choice of the partial integration algorithm. The tail generated by our specific symmetric algorithm will be denoted by $T_m(i_1 \dots i_m)$. The 1 - tail is (unambiguously) given by $T_1(i) = \dot{G}_{ij}$ (i being fixed and j summed over).

With the above abbreviations, the result for Q_5 obtained by an application of the symmetric algorithm can be written as follows,

$$
Q_5 = q_5^5 + q_5^4 + q_5^3 + q_5^2 - q_5^{32} - q_5^{22}
\tag{C.8}
$$

where

$$
\begin{aligned}
q_5^5 &= \dot{G}(12345) + 11 \text{ permutations} \\
q_5^4 &= \dot{G}(1234)\dot{G}_{5i} + 14 \text{ perm.} \\
q_5^3 &= \dot{G}(123)\Bigl\{\dot{G}_{4i}\dot{G}_{5j} + \frac{1}{2}\underline{\dot{G}}_{45}\Bigl[\dot{\not{G}}_{4i} - \dot{\not{G}}_{5i}\Bigr]\Bigr\} + 9 \text{ perm.} \\
q_5^2 &= \dot{G}(12)\Biggl\{\dot{G}_{3i}\dot{G}_{4j}\dot{G}_{5k} + \frac{1}{2}\underline{\dot{G}}_{34}\Bigl[\dot{G}_{5k}\Bigl[\dot{\not{G}}_{3i} - \dot{\not{G}}_{4i}\Bigr] + \dot{\not{G}}_{5i}\Bigl[\dot{G}_{53} - \dot{G}_{54}\Bigr]\Bigr] \\
&\qquad + \frac{1}{2}\underline{\dot{G}}_{35}\Bigl[\dot{G}_{4k}\Bigl[\dot{\not{G}}_{3i} - \dot{\not{G}}_{5i}\Bigr] + \dot{\not{G}}_{4i}\Bigl[\dot{G}_{43} - \dot{G}_{45}\Bigr]\Bigr] \\
&\qquad + \frac{1}{2}\underline{\dot{G}}_{45}\Bigl[\dot{G}_{3k}\Bigl[\dot{\not{G}}_{4i} - \dot{\not{G}}_{5i}\Bigr] + \dot{\not{G}}_{3i}\Bigl[\dot{G}_{34} - \dot{G}_{35}\Bigr]\Bigr]\Biggr\} + 9 \text{ perm.} \\
q_5^{32} &= \dot{G}(123)\dot{G}(45) + 9 \text{ perm.} \\
q_5^{22} &= \dot{G}(12)\dot{G}(34)\dot{G}_{5i} + 14 \text{ perm.}
\end{aligned}
\tag{C.9}
$$

Again we have an over-counting; q_5^{32} is contained once in both q_5^3 and q_5^2, and q_5^{22} is contained twice in q_5^2.

Comparing the 2 - and 3 - tails appearing in (C.9) with our earlier results (4.72),(4.74) for Q_2 and Q_3, we note that there is a simple relation: The tail T_i can be obtained from Q_i, in its un-decomposed form, by rewriting Q_i in the tail variables, and then extending the range of all dummy indices to run over the complete set of variables $\tau_1, \dots, \tau_5$. It is not difficult to

see that this relation generalizes to an arbitrary Q_m, T_m. Consider (the unpermuted term of) q_N^2, which has a 2 - cycle $\dot{G}(12)$ and a tail $T_{N-2}(3\dots N)$. It suffices to consider those terms in q_N^2 having a $\varepsilon_1 \cdot k_2 \varepsilon_2 \cdot k_1$ as their $Z_2(12)$ – component. From the master formula eq.(1.18) one infers that for this part of q_N^2 the partial integration procedure can have involved only partial integrations over the tail variables $\tau_3, \dots, \tau_N$. Thus the calculation of T_{N-2} and the lower order calculation of Q_{N-2} are identical as far as the tail indices are concerned. The presence of the cycle variables for the tail makes itself felt only through an extension of the momentum sums in the master formula, leading to the stated extension rule for dummy indices. The same type of argument shows that the structure of T_m does not depend on the number and lengths of the cycles it multiplies.

At this point it should be noted that every term in Q_N must have at least one cycle factor (this is a combinatorial consequence of the fact that each such term contains a total of $2N$ indices, of which only N are different). Thus the maximal tail occurring in Q_N has length $N-2$. The above connection between T_N and Q_N thus allows us to write down, without going through the partial integration procedure again, Q_6 as follows,

$$Q_6 = q_6^6 + q_6^5 + q_6^4 + q_6^3 + q_6^2 - q_6^{42} - q_6^{33} - q_6^{32} - q_6^{22} + q_6^{222} \tag{C.10}$$

where

$$\begin{aligned}
q_6^6 &= \dot{G}(123456) + \text{permutations} \quad \Bigl(\frac{5!}{2} = 60 \text{ in total}\Bigr)\\
q_6^5 &= \dot{G}(12345)T_1(6) + \text{perm.} \quad \Bigl(\frac{4!}{2}\binom{6}{1} = 72 \text{ in total}\Bigr)\\
q_6^4 &= \dot{G}(1234)T_2(56) + \text{perm.} \quad \Bigl(45 \text{ in total}\Bigr)\\
q_6^3 &= \dot{G}(123)T_3(456) + \text{perm.} \quad \Bigl(20 \text{ in total}\Bigr)\\
q_6^2 &= \dot{G}(12)T_4(3456) + \text{perm.} \quad \Bigl(15 \text{ in total}\Bigr)\\
q_6^{42} &= \dot{G}(1234)\dot{G}(56) + \text{perm.} \quad \Bigl(45 \text{ in total}\Bigr)\\
q_6^{33} &= \dot{G}(123)\dot{G}(456) + \text{perm.} \quad \Bigl(10 \text{ in total}\Bigr)\\
q_6^{32} &= \dot{G}(123)\dot{G}(45)T_1(6) + \text{perm.} \quad \Bigl(60 \text{ in total}\Bigr)\\
q_6^{22} &= \dot{G}(12)\dot{G}(34)T_2(56) + \text{perm.} \quad \Bigl(45 \text{ in total}\Bigr)\\
q_6^{222} &= \dot{G}(12)\dot{G}(34)\dot{G}(56) + \text{perm.} \quad \Bigl(15 \text{ in total}\Bigr)
\end{aligned} \tag{C.11}$$

Here the only new ingredient, T_4, according to the above is related to the un-decomposed Q_4 of eq.(4.77) simply by a relabelling, and an extension of the range of all dummy indices to run from 1 to 6:

$$\begin{aligned}
T_4(abcd) \;=\; & \dot{G}_{ai}\dot{G}_{bj}\dot{G}_{ck}\dot{G}_{dl} + \Bigl\{\frac{1}{2}\underline{\dot{G}}_{ab}\Bigl\{\dot{G}_{ci}\dot{G}_{dj}(\not{\dot{G}}_{ak} - \not{\dot{G}}_{bk}) + \Bigl[\dot{G}_{ci}(\dot{G}_{da} - \dot{G}_{db})\not{\dot{G}}_{dk} + (c \leftrightarrow d)\Bigr]\\
& + \Bigl[(\dot{G}_{ca} - \dot{G}_{cb})\dot{G}_{dc}\not{\dot{G}}_{dk} + (c \leftrightarrow d)\Bigr]\Bigr\} + 5 \quad \text{perm.}\Bigr\} + \Bigl\{\frac{1}{4}\underline{\dot{G}}_{ab}\underline{\dot{G}}_{cd}\Bigl\{\Bigl[\not{\dot{G}}_{ai} - \not{\dot{G}}_{bi}\Bigr]\Bigl[\not{\dot{G}}_{cj} - \not{\dot{G}}_{dk}\Bigr]\\
& + \frac{1}{2}\Bigl[\not{\dot{G}}_{ac} - \not{\dot{G}}_{bc} - \not{\dot{G}}_{ad} + \not{\dot{G}}_{bd}\Bigr]\Bigl[\not{\dot{G}}_{ai} + \not{\dot{G}}_{bi} - \not{\dot{G}}_{ci} - \not{\dot{G}}_{di}\Bigr]\Bigr\} + 2 \text{ perm.}\Bigr\}
\end{aligned} \tag{C.12}$$

Note that the integrand is not yet quite suitable for the application of the cycle substitution rules, since the tails still contain cycles. For this purpose, one should further rewrite Q_6 as [39]

$$Q_6 = Q_6^6 + Q_6^5 + Q_6^4 + Q_6^3 + Q_6^2 + Q_6^{42} + Q_6^{33} + Q_6^{32} + Q_6^{22} + Q_6^{222} \tag{C.13}$$

where, by definition, $Q_6^{(\cdots)}$ is obtained from the corresponding $q_6^{(\cdots)}$ by restricting the range of the dummy indices appearing in its tail so as to precisely eliminate all additional cycles. This also removes the over-counting, so that now all coefficients are unity.

It is now obvious that in this way one arrives at a canonical permutation symmetric version of the Bern-Kosower integrand for the one-loop N - photon/gluon amplitude. Moreover, in [147] it was shown that the cycle decomposition has the additional advantage of constituting a maximal gauge invariant decomposition of this amplitude.

[39] When comparing with [147] note that our present definition of $q_N^{(\cdot)}$ ($Q_N^{(\cdot)}$) there corresponds to $Q_N^{(\cdot)}$ ($\hat{Q}_N^{(\cdot)}$).

Appendix D. Proof of the Cycle Replacement Rule

Combining the results of the previous appendix with the worldline superfield formalism we are now in a position to give a direct and simple proof of the basic cycle replacement rule (2.15) connecting the scalar and fermion loop integrands for the N – photon/gluon amplitudes.

Expanding out the superfield master formula (4.32) for the fermion loop we obtain a formula isomorphic to the scalar case,

$$
\begin{aligned}
\exp\biggl\{\sum_{i,j=1}^{N}\Bigl[\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j + iD_i\hat{G}_{ij}\varepsilon_i\cdot k_j + \frac{1}{2}D_iD_j\hat{G}_{ij}\varepsilon_i\cdot\varepsilon_j\Bigr]\biggr\}\,|_{\varepsilon_1\dots\varepsilon_N} \quad &= \\
(-i)^n P_N(-D_i\hat{G}_{ij}, D_iD_j\hat{G}_{ij})\exp\biggl\{\sum_{i,j=1}^{N}\frac{1}{2}\hat{G}_{ij}k_i\cdot k_j\biggr\} & \qquad \text{(D.1)}
\end{aligned}
$$

Here P_N is the same polynomial which appears in the expansion (4.68) of the ordinary master formula. We now apply to this integrand the same partial algorithm as in the previous appendix, just with ordinary derivatives replaced by super derivatives. In this way we can remove all second derivatives $D_iD_j\hat{G}_{ij}$. A recursive analysis analogous to the one performed in appendix C shows, that the final result of the partial integrations is almost isomophic to the ordinary Q_N, except that we have to take into account that $D_i\hat{G}_{ij} \neq -D_j\hat{G}_{ij}$. This means that, for example, the formula for the ordinary 2 - tail

$$
T_2(ab) \quad = \quad \dot{G}_{ai}\dot{G}_{bj} + \frac{1}{2}\underline{\dot{G}}_{ab}(\cancel{\dot{G}}_{ai} - \cancel{\dot{G}}_{bi}) \qquad \text{(D.2)}
$$

in the super case has to be written differently,

$$
\hat{T}_2(ab) \quad = \quad D_a\hat{G}_{ai}\varepsilon_a\cdot k_iD_b\hat{G}_{bj}\varepsilon_b\cdot k_j + \frac{1}{2}\Bigl[D_a\hat{G}_{ab}\varepsilon_a\cdot\varepsilon_bD_b\hat{G}_{bi}k_b\cdot k_i + (a\leftrightarrow b)\Bigr] \qquad \text{(D.3)}
$$

The general structure found above remains the same, i.e. the final integrand Q_N can be decomposed into a sum of terms $Q_N^{(\dots)}$ which individually are products of bi-cycles and tails, where the tails do not contain closed cycles of indices. Each term contains precisely N factors of $D_i\hat{G}_{ij}$'s. Now, observe that, since in the master formula (4.32) D_i appears only together with ε_i, all terms in the original integrand P_N contain every D_i precisely once. Since this property is preserved by the partial integrations, and the cycles contain only D_i's in the cycle variables, it follows that the tails can contain only D_i's in the tail variables. From

$$
D_i\hat{G}_{ij} \quad = \quad \theta_i\dot{G}_{Bij} - \theta_jG_{Fij} \qquad \text{(D.4)}
$$

it is then clear that a G_{Fij} produced by a $D_i\hat{G}_{ij}$ in a tail cannot survive the Grassmann integrations if one of the indices i,j is a cycle index; terms of this kind have too many cycle - θ's and too few tail - θ's. Therefore after the θ - integrations for all surviving G_{Fij}'s the indices i,j are either both cycle variables, or both tail variables. Since from the structure of the

component field integral $\mathcal{D}\psi$ it is clear that G_{Fij}'s can generally only appear in cycles, G_{Fij}'s from tails would therefore have to form cycles among themselves; but this is not possible, since the tails do not contain closed chains of indices. We conclude that, in fact, all G_{Fij}'s coming from tails must drop out in the θ - integrations.

This leaves us with the basic super cycle integrals, for which we can directly verify that

$$\begin{aligned}\int d\theta_{i_1}\cdots\theta_{i_n} D_{i_1}\hat{G}_{i_1i_2}D_{i_2}\hat{G}_{i_2i_3}\cdots D_{i_n}\hat{G}_{i_ni_1} &= (-1)^{\frac{n(n+1)}{2}}\Bigl(\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}\\ &\quad -G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}\Bigr)\end{aligned} \tag{D.5}$$

Appendix E. Massless 1 – Loop 4 – Point Tensor Integrals

In the worldline parametrization the massless scalar box becomes

$$
\begin{aligned}
B[k_1,k_2,k_3,k_4] &= \int_0^\infty \frac{dT}{T}T^{4-\frac{D}{2}}\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\,\exp\Bigl\{T\sum_{i<j}G_{Bij}k_i\cdot k_j\Bigr\}\\
&= \Gamma(2-\frac{\epsilon}{2})\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3\frac{1}{\Bigl[-\sum_{i<j}G_{Bij}k_i\cdot k_j\Bigr]^{2-\frac{\epsilon}{2}}}
\end{aligned}
\tag{E.1}
$$

This is essentially eq.(4.14) specialized to $N=4$. We have already rescaled to the unit circle and put $u_4=0$. We return to Feynman parameters a_i via eq.(4.15),

$$
a_1 = 1-u_1,\quad a_2 = u_1-u_2,\quad a_3 = u_2-u_3,\quad a_4 = u_3 \tag{E.2}
$$

($\alpha_i = Ta_i$) so that

$$
\int_0^1 du_1\int_0^{u_1}du_2\int_0^{u_2}du_3 = \int_0^1 da_1da_2da_3da_4\delta(1-\sum_{i=1}^4 a_i) \tag{E.3}
$$

Also we introduce the Mandelstam variables

$$
s=(k_1+k_2)^2,\quad t=(k_2+k_3)^2 \tag{E.4}
$$

With all external legs massless and on-shell, $k_i^2=0$, the Feynman denominator simplifies to

$$
\sum_{i<j}G_{Bij}k_i\cdot k_j = -a_1a_3s - a_2a_4t \tag{E.5}
$$

Generally we will also have a numerator polynomial $P(\{a_i\})$. Let us thus define

$$
I[P(\{a_i\})] \equiv \Gamma(2-\frac{\epsilon}{2})\int_0^1\prod_{i=1}^4 da_i\,\delta\Bigl(1-\sum_{i=1}^4 a_i\Bigr)\frac{P(\{a_i\})}{\Bigl[a_1a_3s+a_2a_4t\Bigr]^{2-\frac{\epsilon}{2}}} \tag{E.6}
$$

In the rest of this subsection we will perform some manipulations which allow us to generate the numerator polynomial by differentiations performed on the denominator polynomial.

First it is useful to homogeneize the denominator by the following well-known transformation due to 't Hooft and Veltman [278],

$$
\begin{aligned}
a_i &= \frac{\alpha_i x_i}{\sum_{j=1}^4\alpha_j x_j} &\qquad i=1,2,3\\
a_4 &= \frac{\alpha_4(1-x_1-x_2-x_3)}{\sum_{j=1}^4\alpha_j x_j}
\end{aligned}
\tag{E.7}
$$

with the constraints

$$\alpha_1\alpha_3 = \frac{1}{s}, \quad \alpha_2\alpha_4 = \frac{1}{t} \tag{E.8}$$

Note that this implies

$$\sum_{j=1}^{4} \frac{a_j}{\alpha_j} = \frac{1}{\sum_{j=1}^{4} \alpha_j x_j} \tag{E.9}$$

This transformation leads to

$$I[P(\{a_i\})] = \Gamma(2-\frac{\epsilon}{2}) \int_0^1 \prod_{i=1}^{4} dx_i\, \delta\Big(1 - \sum_{j=1}^{4} x_j\Big) \frac{\Big(\prod_{i=1}^{4} \alpha_i\Big)(\sum_{j=1}^{4} \alpha_j x_j)^{-\epsilon} P(\{a_i\})}{\Big[x_1x_3 + x_2x_4\Big]^{2-\frac{\epsilon}{2}}} \tag{E.10}$$

If we now specialize P to be a polynomial P_m of definite degree m, this can be further rewritten as

$$\begin{aligned} I[P] &= \Big(\prod_{i=1}^{4} \alpha_i\Big)\Gamma(2-\frac{\epsilon}{2}) \int_0^1 \prod_{i=1}^{4} dx_i\, \delta\Big(1 - \sum x_j\Big) \frac{P_m(\{\alpha_i x_i\})\Big(\sum_{j=1}^{4} \alpha_j x_j\Big)^{-m-\epsilon}}{\Big[x_1x_3 + x_2x_4\Big]^{2-\frac{\epsilon}{2}}} \\ &= \Big(\prod_{i=1}^{4} \alpha_i\Big)\Gamma(2-\frac{\epsilon}{2}) \frac{\Gamma(1-\epsilon-m)}{\Gamma(1-\epsilon)} \int_0^1 \prod_{i=1}^{4} dx_i\, \delta(1 - \sum x_j) \frac{P_m(\{\alpha_i \frac{\partial}{\partial\alpha_i}\})\Big(\sum_{j=1}^{4} \alpha_j x_j\Big)^{-\epsilon}}{\Big[x_1x_3 + x_2x_4\Big]^{2-\frac{\epsilon}{2}}} \\ &= \frac{\Gamma(1-\epsilon-m)}{\Gamma(1-\epsilon)} \Big(\prod_{i=1}^{4} \alpha_i\Big) P_m(\{\alpha_i \frac{\partial}{\partial\alpha_i}\}) \Big(\frac{I[1]}{\prod_{i=1}^{4} \alpha_i}\Big) \end{aligned} \tag{E.11}$$

Here it is understood that

$$\Big(\alpha_i \frac{\partial}{\partial\alpha_i}\Big)^n \equiv \alpha_i^n \Big(\frac{\partial}{\partial\alpha_i}\Big)^n \tag{E.12}$$

It remains to calculate $I[1]$,

$$\begin{aligned} I[1] &= \Big(\prod_{i=1}^{4} \alpha_i\Big)\Gamma(2-\frac{\epsilon}{2}) \int_0^1 d^4x\, \delta(1 - \sum x_j) \frac{\Big(\sum_{j=1}^{4} \alpha_j x_j\Big)^{-m-\epsilon}}{\Big[x_1x_3 + x_2x_4\Big]^{2-\frac{\epsilon}{2}}} \\ &= \Gamma(2-\frac{\epsilon}{2}) \int_0^1 d^4a\, \delta(1 - \sum a_i) \frac{1}{\Big[a_1a_3s + a_2a_4t\Big]^{2-\frac{\epsilon}{2}}} \end{aligned} \tag{E.13}$$

This can be easily done using the following transformation of variables going back to Karplus and Neuman [279],

$$a_1 = (1-y)(1-z), \quad a_2 = z(1-y), \quad a_3 = y(1-x), \quad a_4 = xy \tag{E.14}$$

which has a Jacobian

$$|\frac{\partial(a_1,a_2,a_3)}{\partial(x,y,z)}| = y(1-y) \tag{E.15}$$

This transformation leads to

$$I[1] = \Gamma(2-\frac{\epsilon}{2})\int_0^1 dx \int_0^1 dy \int_0^1 dz \frac{1}{[y(1-y)]^{1-\frac{\epsilon}{2}}\Big[(1-x)(1-z)s + xzt\Big]^{2-\frac{\epsilon}{2}}} \tag{E.16}$$

The y - integral factors out and gives an Euler Beta function,

$$\int_0^1 dy \Big[y(1-y)\Big]^{-1+\frac{\epsilon}{2}} = B(\frac{\epsilon}{2},\frac{\epsilon}{2}) \tag{E.17}$$

For the remaining double integral the x - integration is elementary, and the resulting z - integral can be expressed in terms of hypergeometric functions using the formula

$$\int_0^1 dt\, t^{a-1}(1-t)^{b-a-1}(1-\lambda t)^{-c} = \frac{\Gamma(a)\Gamma(b-a)}{\Gamma(b)}\,{}_2F_1(a,c;b;\lambda) \tag{E.18}$$

In this way one arrives at

$$I[1] = \frac{8}{\epsilon^2} r_\Gamma s^{-1}\Big\{t^{\frac{\epsilon}{2}-1}{}_2F_1(\frac{\epsilon}{2},1;1+\frac{\epsilon}{2};1+\frac{t}{s}) - s^{\frac{\epsilon}{2}-1}{}_2F_1(1,1;1+\frac{\epsilon}{2};1+\frac{t}{s})\Big\} \tag{E.19}$$

where

$$r_\Gamma \equiv \frac{\Gamma(1-\frac{\epsilon}{2})\Gamma^2(1+\frac{\epsilon}{2})}{\Gamma(1+\epsilon)} \tag{E.20}$$

The ϵ - expansion of this expression is

$$I[1] = r_\Gamma \alpha_1\alpha_2\alpha_3\alpha_4 \Big[\frac{8}{\epsilon^2}\Big((\alpha_1\alpha_3)^{-\frac{\epsilon}{2}} + (\alpha_2\alpha_4)^{-\frac{\epsilon}{2}}\Big) - \ln^2\Big(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\Big) - \pi^2\Big] + \mathrm{O}(\epsilon) \tag{E.21}$$

This is sufficient as far as the naked box integral is concerned, but not if used in formula (E.11), since for nontrivial P the factor $\Gamma(1-\epsilon-m)$ in front has a pole, so that $I[1]$ here would be needed to order $\mathrm{O}(\epsilon)$. Rather than using this formula as it stands, it is simpler to start the recursion with the four polynomials of degree $m=1$. In this case there is no singular prefactor, and it suffices to give those polynomials to order ϵ^0:

$$\begin{aligned} I[a_1] &= I[a_3] = r_\Gamma\alpha_1\alpha_2\alpha_3\alpha_4\Big\{\frac{4}{\epsilon^2}(\alpha_2\alpha_4)^{-\frac{\epsilon}{2}} - \frac{\alpha_1\alpha_3}{2(\alpha_1\alpha_3+\alpha_2\alpha_4)}\Big[\ln^2\Big(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\Big)+\pi^2\Big]\Big\} + \mathrm{O}(\epsilon) \\ I[a_2] &= I[a_4] = r_\Gamma\alpha_1\alpha_2\alpha_3\alpha_4\Big\{\frac{4}{\epsilon^2}(\alpha_1\alpha_3)^{-\frac{\epsilon}{2}} - \frac{\alpha_2\alpha_4}{2(\alpha_1\alpha_3+\alpha_2\alpha_4)}\Big[\ln^2\Big(\frac{\alpha_1\alpha_3}{\alpha_2\alpha_4}\Big)+\pi^2\Big]\Big\} + \mathrm{O}(\epsilon) \end{aligned} \tag{E.22}$$

Appendix F. Some Worldloop Formulas

In this appendix we abbreviate $G_B = G$, $\mathcal{G}_B = \mathcal{G}$. All formulas are written for the unit circle, $T = 1$.

Chain Integrals involving $\dot{G}, G_F$

$$
\begin{aligned}
\int_0^1 du_2 \dots du_n \dot{G}_{12}\dot{G}_{23}\dots\dot{G}_{n(n+1)} &= 2^n < u_1 \mid \partial_P^{-n} \mid u_{n+1} > \\
&= -\frac{2^n}{n!}B_n(|u_1 - u_{n+1}|)\mathrm{sign}^n(u_1 - u_{n+1}) \qquad \text{(F.1)}\\
\int_0^1 du_2 \dots du_n G_{F12}G_{F23}\dots G_{Fn(n+1)} &= 2^n < u_1 \mid \partial_A^{-n} \mid u_{n+1} > \\
&= \frac{2^{n-1}}{(n-1)!}E_{n-1}(|u_1 - u_{n+1}|)\mathrm{sign}^n(u_1 - u_{n+1})
\end{aligned}
\tag{F.2}
$$

Here $B_n(x)$ ($E_n(x)$) denotes the n^{th} Bernoulli (Euler) polynomial. The right hand sides can be rewritten in term of $G, \dot{G}, G_F$:

$$< 1 \mid \partial_P^{-1} \mid 2 > = \frac{1}{2}\dot{G}_{12} \tag{F.3}$$

$$< 1 \mid \partial_P^{-2} \mid 2 > = \frac{1}{2}(G_{12} - \frac{1}{6}) \tag{F.4}$$

$$< 1 \mid \partial_P^{-3} \mid 2 > = -\frac{1}{12}\dot{G}_{12}G_{12} \tag{F.5}$$

$$< 1 \mid \partial_P^{-4} \mid 2 > = -\frac{1}{24}(G_{12}^2 - \frac{1}{30}) \tag{F.6}$$

$$< 1 \mid \partial_P^{-5} \mid 2 > = \frac{1}{5!}\dot{G}_{12}(\frac{1}{2}G_{12}^2 + \frac{1}{6}G_{12}) \tag{F.7}$$

$$< 1 \mid \partial_P^{-6} \mid 2 > = \frac{1}{6!}(G_{12}^3 + \frac{1}{2}G_{12}^2 - \frac{1}{42}) \tag{F.8}$$

$$< 1 \mid \partial_A^{-1} \mid 2 > = \frac{1}{2}G_{F12} \tag{F.9}$$

$$< 1 \mid \partial_A^{-2} \mid 2 > = -\frac{1}{4}G_{F12}\dot{G}_{12} \tag{F.10}$$

$$< 1 \mid \partial_A^{-3} \mid 2 > = -\frac{1}{4}G_{F12}G_{12} \tag{F.11}$$

$$< 1 \mid \partial_A^{-4} \mid 2 > = \frac{1}{24}G_{F12}\dot{G}_{12}(G_{12} + \frac{1}{2}) \tag{F.12}$$

etcetera.

Chain Integrals involving G

$$\int_0^1 du_2\, G_{12}G_{23} = -\frac{1}{6}G_{13}^2 + \frac{1}{30} \tag{F.13}$$

$$\int_0^1 du_2 \int_0^1 du_3\, G_{12}G_{23}G_{34} = \frac{1}{90}(G_{14}^3 + \frac{1}{2}G_{14}^2) + \frac{1}{180} - \frac{6}{7!} \tag{F.14}$$

Chain Integrals involving $\dot{\mathcal{G}}$

$$\int_0^1 du\, \dot{\mathcal{G}}_{1u}\dot{\mathcal{G}}_{u2} = -i\frac{\partial}{\partial \mathcal{Z}}\dot{\mathcal{G}}_{12} = -\frac{\mathrm{e}^{-i\mathcal{Z}\dot{G}_{12}}}{\sin^2(\mathcal{Z})}\Big[\cos(\mathcal{Z}) + i\dot{G}_{12}\sin(\mathcal{Z})\Big] + \frac{1}{\mathcal{Z}^2} \tag{F.15}$$

Integrals appearing in the Calculation of the 2-Loop Spinor QED β - Function

$$\int_0^1 du_1 \int_0^1 du_2 (\dot{G}_{12}^2 - G_{F12}^2) = -\frac{2}{3} \tag{F.16}$$

$$\int_0^1 du_1 (G_{13} - G_{14})^2 = \frac{1}{3}G_{34}^2 \tag{F.17}$$

$$\begin{aligned}\int_0^1 du_1 \int_0^1 du_2 (\dot{G}_{12}\dot{G}_{23}\dot{G}_{34}\dot{G}_{41} - &\\ -G_{F12}G_{F23}G_{F34}G_{F41}) &= -\frac{8}{3}G_{34}^2 - \frac{4}{3}G_{34}\end{aligned} \tag{F.18}$$

$$\begin{aligned}\int_0^1 du_1 \int_0^1 du_2 (\dot{G}_{13}\dot{G}_{32}\dot{G}_{24}\dot{G}_{41} - &\\ -G_{F13}G_{F32}G_{F24}G_{F41}) &= 4G_{34}^2 + \frac{8}{3}G_{34} - \frac{8}{9}\end{aligned} \tag{F.19}$$

2-Point Integrals

$$\int_0^1 du [G(u,u_1) - G(u,u_2)]^{2n} = \frac{G(u_1,u_2)^{2n}}{2n+1} \qquad (n \in \mathbf{N}) \tag{F.20}$$

$$\int_0^1 du \exp\{c[G(u,u_1) - G(u,u_2)]\} = \frac{\sinh[cG(u_1,u_2)]}{cG(u_1,u_2)} \tag{F.21}$$

$$\int_0^1 du\, \dot{G}^k(u_1,u)\dot{G}^l(u,u_2) = \frac{k!l!}{2}\sum_{i=0}^{l} \frac{\Big(1 - (-1)^{k+l-i+1}\Big)\dot{G}_{12}^i - \Big(1 - (-1)^i\Big)\dot{G}_{12}^{k+l-i+1}}{i!(k+l-i+1)!} \tag{F.22}$$

3-Point Integrals

$$
\begin{aligned}
\int_0^1 du\, \dot{G}(u,u_1)\dot{G}(u,u_2)\dot{G}(u,u_3) &= -\frac{2}{3}\{\dot{G}_{23}[G_{12}-G_{13}]+\dot{G}_{31}[G_{23}-G_{21}]+\dot{G}_{12}[G_{31}-G_{32}]\} \\
&= -\frac{1}{6}(\dot{G}_{12}-\dot{G}_{23})(\dot{G}_{23}-\dot{G}_{31})(\dot{G}_{31}-\dot{G}_{12})
\end{aligned}
\tag{F.23}
$$

***n*-Point Integrals**

$$
\int_0^1 du\, \mathrm{e}^{\sum_{i=1}^n c_i \dot{G}(u,u_i)} = \frac{\sum_{i=1}^n \sinh(c_i)\, \mathrm{e}^{\sum_{j=1}^n c_j \dot{G}_{ij}}}{\sum_{i=1}^n c_i}
\tag{F.24}
$$

Miscellaneous Identities

$$
\dot{G}_{ij}^2 = 1-4G_{ij} \tag{F.25}
$$

$$
\dot{G}_{12}+\dot{G}_{23}+\dot{G}_{31} = -G_{F12}G_{F23}G_{F31} \tag{F.26}
$$

$$
\frac{d^n}{du_i^n}(G_{ij}^n) = n! P_n(\dot{G}_{ij}) \tag{F.27}
$$

where P_n denotes the n^{th} Legendre polynomial.